Amazing Locus

Teacher Notes and Answers

TI-Nspire™   Investigation   Student   50 min

Teacher Notes:

The problem starts with a simple statement:

> " Find the equation to the locus formed by the intersection of the altitudes of a triangle with vertices on the function $y = \frac{1}{x}$ "

The problem sounds complicated, but its not. The algebra sounds complicated, but its not, yet most students struggle to make a start on this and similar problems. The solution is elegant, the result delightful and the opportunities to further explore simply wonderful.

Before giving students the worksheet, put the problem statement on the board. Ask students to write down any terms they do not understand. Some students may highlight the entire question, challenge these students or lighten the moment by asking them to define words like "triangle". [Given they highlighted the entire question.] Here we are trying to identify the key terms that create the cerebral hurdles students erect that prevent them from taking the first cognitive step.

Most commonly highlighted words:

- Altitudes
- Vertices
- Locus

Altitude: This word is familiar; the context is not. Identify where they have heard the word previously: 'height of a mountain' or 'height of a plane'. A couple of diagrams help clarify the definition. Draw a simple 'triangular' shaped mountain with a line passing through the 'base' and the 'summit'. Do not draw this line at right angles to the base. Students quickly correct this error and note that the altitude must be perpendicular to the base and eventually arrive at a more formal definition: "the altitude is a line perpendicular to the base passing through the opposite vertex".

Vertices: This word is less familiar in everyday language, but can be compared with a more common term: "corner". Associating a more familiar word helps build upon existing connections, with an understanding that mathematical terms may be more rigorous in their definition. A perfect example here is the common term 'average' which is often (mistakably) taken to be equivalent to the mathematical term: mean, which of course refers to a specific calculation.

Locus: This may be a completely new term for students. The formal definition is not initially helpful to most students: "A point, line, or surface moving according to mathematically defined conditions". Getting students to 'model' a locus works well.

- Students line up across the front of the classroom.
- Place an object in the middle of the room.
- Students walk away from the front of the classroom towards the rear (perpendicular to the front of the room)
- Students stop when they are the same distance (equidistant) from the front of the room as they are from the object. [See diagram]

Authors: Peter Fox

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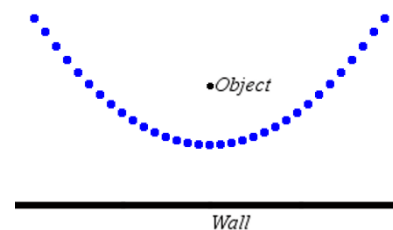
Students should now be standing on a curve. Each student's location has been defined by the condition: 'equidistant from a line (wall) and a point (object)'. The curve is a locus. This particular locus has a familiar equation, a parabola.

With an understanding of all the words, students can start breaking down the question and work towards solving the problem. The questions in the worksheet provide the order in which students can tackle the question, so it is left up to the teacher as to whether this becomes a class exercise or a guided series of questions (worksheet) for students to complete.

Exploring the Problem with Help from TI-Nspire

Start a new document and insert a Graphs Application.

We can break the problem statement into pieces, tackling the things that we know first. The problem talks about a "triangle with vertices on the function…", so before we draw the triangle we must attend to the function.
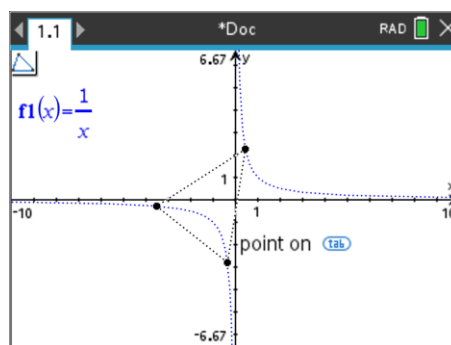
Graph the function: $y = \dfrac{1}{x}$
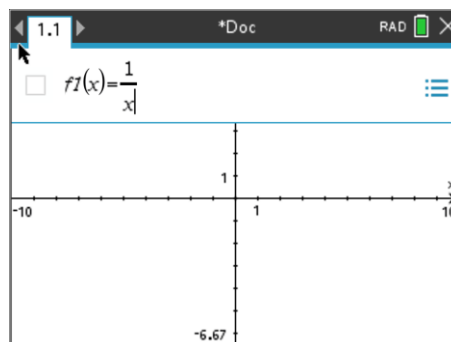
The next step in the problem is to draw the "triangle with vertices on the function …"

[Menu] > Geometry > Shapes > Triangle

The triangle tool appears on the top left corner of the screen. When this is active the mouse/cursor turns into a pencil. Move the pencil over the function until it says "point on", then click to place the vertex on the function.


Place all three vertices on the function, then press [esc] to release the triangle tool.

Now we have a triangle with vertices on the function, we need to add in the altitudes. There are three altitudes. Each altitude passes through a vertex and is perpendicular to the opposite side.

[Menu] > Geometry > Construction > Perpendicular

With the perpendicular tool selected, once again the mouse/cursor changes to a pencil.

Click on a vertex, notice that the tool tip states "Point On", select the vertex then the opposite side, the tool tip will display "Side".
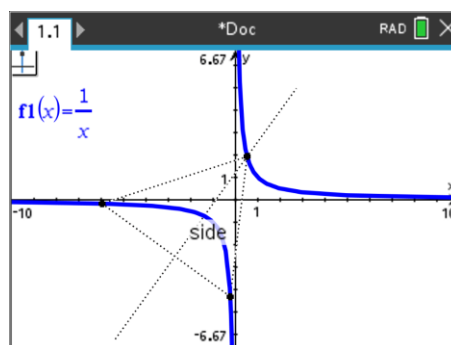
Repeat this process for all three vertices and their subsequent altitudes.

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When several objects are close together on the screen it can be difficult to manually isolate a specific object. Think of the objects as layered. When the pointer is close to a collection of objects, press the **TAB** key to successively identify the nearby objects.

Teacher Notes:

Students may find it difficult to identify the 'triangle' from the altitudes. Students can fill the triangle with colour, change the line attributes. A 'right mouse click' (Ctrl + Menu) on any object reveals the range of options.

At this point students should notice that the three altitudes intersect at one point, furthermore that point 'appears' to be on the function! How cool is that?

Students should be encouraged to drag the vertices of the triangle to see that the point of intersection remains on the function.
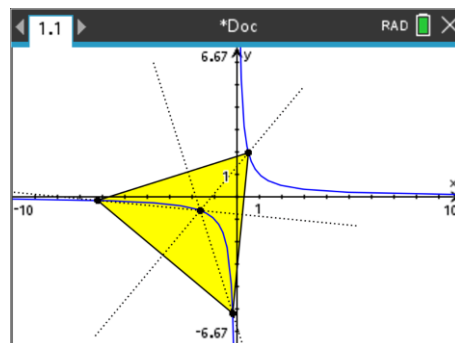
Question: 1.

Where do the altitudes of the triangle appear to intersect?

Answer: The altitudes appear to intersect on the function y = 1/x.

You can add a point of intersection between any two lines by using the "Point of Intersection" tool located in the Points and Lines options in the Geometry menu. On the CX II series, simply press "P", select point and then navigate to the point of intersection.

The following naming system applies to the remaining questions. Vertices have been labelled for easy references and altitudes are named after the vertex they pass through.

For Questions 2 to 6 the coordinates for the triangle's vertices are: A: (1, 1); B: (- ¼, - 4) and C: (- 4, - ¼) .

Question: 2.

i) Determine the gradient of line BC.

Answer: -1

ii) Determine the gradient of altitude A.

Answer: 1 (Perpendicular)

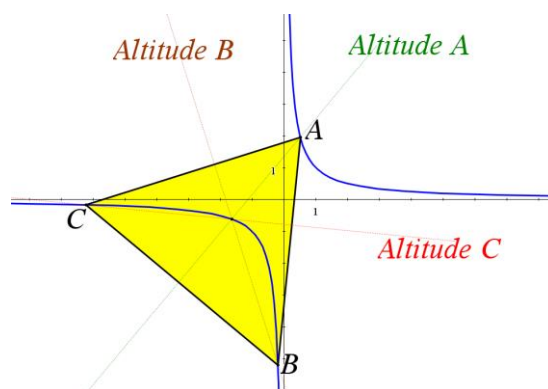iii) Determine the equation to altitude A.

Answer: $y = x$

Question: 3.

Determine the equation for altitude B.

Answer: $y = -4x - 5$

Question: 4.

Determine the equation for altitude C.

Answer: $y = -\dfrac{1}{4}x - \dfrac{5}{4}$

Figure 1 - Diagram not to scale

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Question: 5.

Use simultaneous equations to determine the point of intersection between altitude A and B.

Answer: Note that the simplicity of altitude A, makes substitution the most straight forward solution process.

$y = -4x - 5 \mid y = x$

$x = -4x - 5$ 　　　Which in turn means $y = -1$ making the point of intersection (-1, -1).

$x = -1$

Question: 6.

Determine the point of intersection between altitude B and C and comment on your finding.

Answer: Note that students may use their calculator even though the by-hand process is straight forward:

$-\dfrac{1}{4}x - \dfrac{5}{4} = -4x - 5$

$-x - 5 = -16x - 20$ 　As above, $y = -1$ which results in all three altitudes intersecting at (-1, -1)

$15x = -15$

$x = -1$

Comment(s): All three altitudes intersect at a single point, also the point of intersection is on the original function.

Question: 7.

Create your own set of coordinates for points A, B and C, making sure they are all on the function.
Determine the point of intersection for your three altitudes.

Answer: Each student should have a unique set of coordinates and therefore a unique set of equations. Teachers can check the answers easily using the general equations that are established in Question 8, furthermore, teachers can use the Notes or Graphs Application on TI-Nspire CAS to generate the specific equations for each students selected points.

Question: 8.

Points A, B and C have coordinates: $\left(a, \dfrac{1}{a} \right)$, $\left(b, \dfrac{1}{b} \right)$ and $\left(c, \dfrac{1}{c} \right)$.

Determine the general equations for each altitude and the subsequent point of intersection.
Comment on your findings.

Answer: Altitude A: $y = bc(x - a) + \dfrac{1}{a}$

Note that this equation is easily established from the translational form of a straight line.
Students may recognise the 'pattern' and therefore determine:

Altitude B: $y = ac(x - b) + \dfrac{1}{b}$ & Altitude C: $y = ab(x - c) + \dfrac{1}{c}$

Solving simultaneously: $x = \dfrac{-1}{abc}$ & $y = -abc$

The relationship between x and y is therefore: $y = \dfrac{1}{x}$, that is the altitudes always intersect on the function!

Students should also note the significance of the negative sign. If A, B and C are located in the first quadrant, corresponding to $a > 0$, $b > 0$ and $c > 0$ the point of intersection is such that x < 0 and vice versa.

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Question: 9.

Explore variations of this function to see if the locus continues to follow a distinct path:

i) $y = \dfrac{1}{x-h}$

ii) $y = \dfrac{1}{x} + k$

iii) $y = \dfrac{1}{ax}$

Answer: All three transformations still have the locus of the orthocentre falling on the function.

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