

# Solid of Revolution

## Answers

7 8 9 10 11 12



TI-Nspire™



Activity



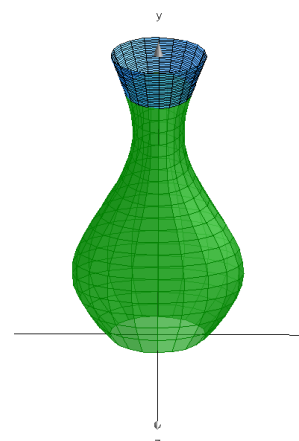
Student



60 min

## Introduction

A liquid flows into an irregular shaped vase at a constant rate, is it possible to describe the changing height of the fluid using mathematics? In this investigation you will collect data for the height of the liquid at various times, produce mathematical equations to model these rates and changes, determine an equation for the volume of the liquid at any height  $h$ , and finally, produce a 3D model of the vase! The 3D model will be dynamic, meaning you will be able to rotate it in any direction.



## Equipment

- YouTube video:
- TI-Nspire Calculator
- TI-Nspire file: Solid

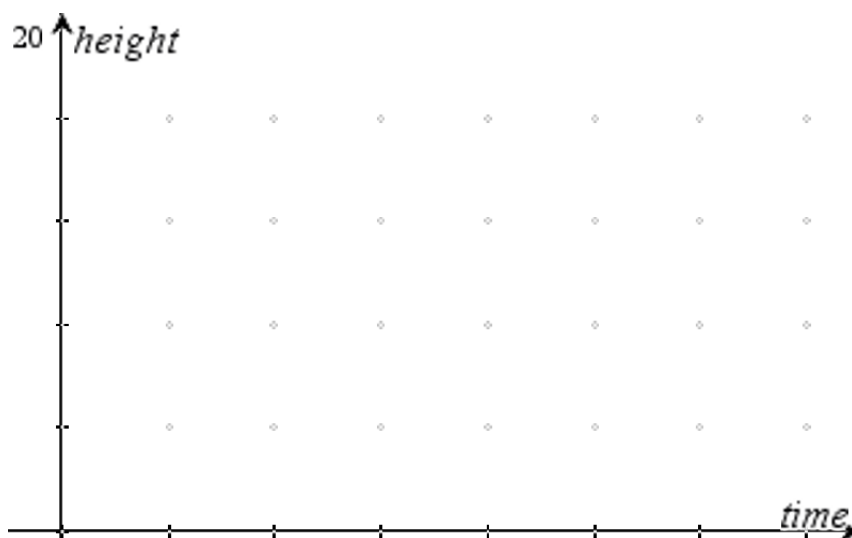
## Filling the Vase

### Question: 1.

Imagine the vase pictured above is being filled at a constant rate to a depth of 20cm (as shown). If this process takes 90 seconds, sketch a graph of the height versus time.

Answers will vary. This question can be used to help gauge student's prior knowledge / understanding of rates.

Note: The "Fill the Vase" activity on the Senior Nspired Curriculum site is a good prerequisite activity.



Open the TI-Nspire Solids of Revolution file. Read through the instructions on Page 1.1 then navigate to the spreadsheet on page 1.2 then watch the YouTube video:

Related Rates and Solid of Revolution

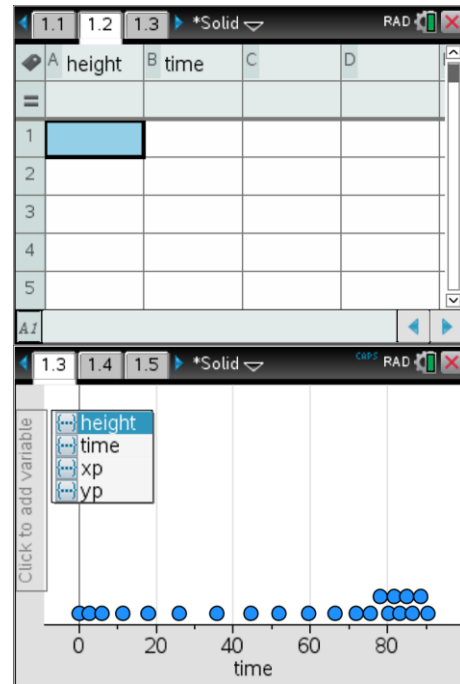
<http://bit.ly/JugFilling>

Enter the data on page 1.2 (Spreadsheet).



Graph your data on page 1.3 of the document.

Although it was the height that was used to determine the time interval, this was due to practical limitations. Time will be plotted on the independent axis and height on the dependent.



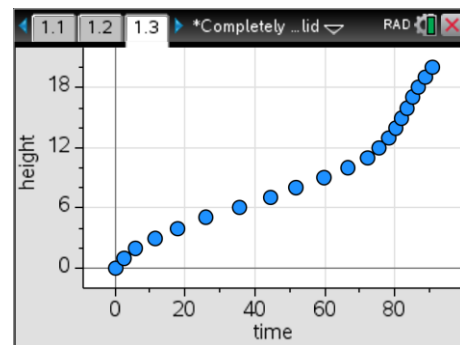
### Question: 2.

Relate key features of the graph to the shape of the vase including a comparison of your sketch graph with the one obtained from the video.

If the video is done as a class, all students should have the same results, sample shown opposite.

#### Key Features:

- Wide regions of vase correspond to smaller gradient on sketched function and data.
- Narrow regions of vase correspond to steeper gradient on sketched function and data.
- Transitions between each region should be smooth (student sketch).



## Volume of the Vase

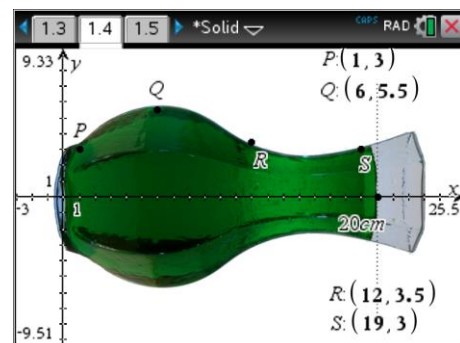
**Note:** Make sure your document settings have Float = 5 or 6. Due to the practical nature of the questions in this section, calculations need a high level of accuracy in order to model the shape of the vase.

Navigate to page 1.4 where a picture of the vase is shown. The aim here is to develop an equation to model the shape of the vase. A cubic function can provide an approximate model.

Four points P, Q, R and S have been plotted and their coordinates stored:

$$P:(x_1, y_1), \quad Q:(x_2, y_2), \quad R:(x_3, y_3) \quad \& \quad S:(x_4, y_4)$$

$$P:(1,3), \quad Q:(6,5.5), \quad R:(12,3.5) \quad \& \quad S:(19,3)$$





The [VAR] key can be used to recall stored variables. This is a very useful way of recalling values stored on other pages within the same problem.

### Question: 3.

Determine the equation for a cubic passing through points P, Q, R and S. Graph it on the same page and discuss how well it fits the shape of the vase.

**Important:** Define your equation as:  $f(x)$

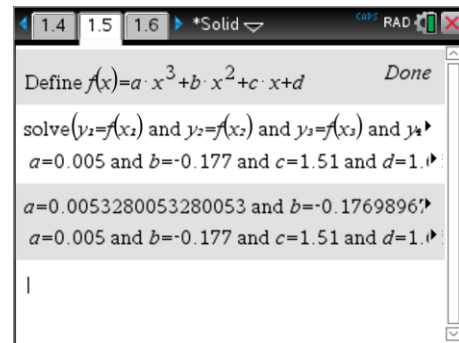
$$f(x) = 0.00533x^3 - 0.177x^2 + 1.510x + 1.662$$

Graph is a relatively good fit. The base of the vase shows a small error, so too the sharper curves where the vase narrows. Most significantly the function is a very poor model beyond the 20cm 'depth' of the vase.

#### Note:

Students can use the stored variables:  $x_1, x_2 \dots$  to determine the equation. The parameters  $\{a, b, c, d\}$  are initially displayed with limited accuracy, copying and pasting the result provides for more accuracy and makes it easier to store each.

Example:  $a := 0.005328$



### Question: 4.

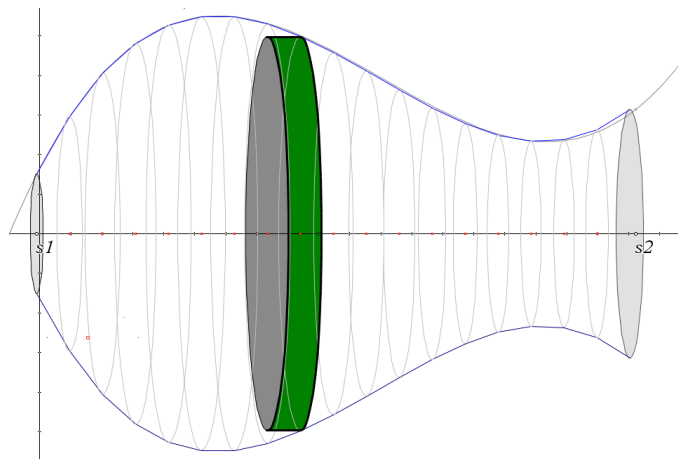
Explain why the vase is easier to model when tipped on its side.

Placing the vase on its side means a polynomial can be used to model the curve. If the vase was placed vertically on the axis it would not be a function, it would have a one to many mapping.

The theoretical volume of the vase can be determined by rotating the function around the  $x$ -axis.

$$V = \pi \int f(x)^2 dx$$

The actual volume of the vase is 1 litre when filled to a depth of 20cm, the amount shown in the original picture.



**Question: 5.**

Use the calculator (CAS) to determine the volume of the vase when filled to a depth of 20cm by rotating the function obtained in Question 3 around the  $x$  axis. Discuss the accuracy of the result.

$$v = \pi \int_0^{20} f(x)^2 dx \approx 999$$

This result compares favourably with the actual...  $1000\text{cm}^3 = 1000 \text{ ml} = 1 \text{ litre}$ .

**Question: 6.**

Use the calculator (CAS) to determine an expression for the volume of liquid in the vase as a function of height ( $h$ ).

Define this function as  $v(h)$

$$v(h) = 1.274 \times 10^{-5} h \cdot (h^6 - 77.51h^5 + 2338h^4 - 31855h^3 + 139017h^2 + 618705h + 680998)$$

**Question: 7.**

According to the function:  $v(h)$ , what would be the height of the liquid in the vase when it is half full? (500ml). Compare this answer with the data collected in Question 1.

*solve*( $v(h) = 500, h$ ) This produces a result of  $h = 7.28$  which compares favourably with the data.

Since the volume is changing at a constant rate, the vase should be half full when half the time has expired. Total time to fill vase: 90 seconds, which of course means the vase should be half full at 45s. The data shows that  $h \approx 7\text{cm}$  when  $t \approx 44\text{s}$ . When  $h \approx 8\text{cm}$ ,  $t \approx 51\text{s}$  so it follows that at  $t = 45\text{s}$  the height will be between 7cm and 8cm... but closer to 7cm, as predicted from the equation solution.

**Related Rates**

The aim of this section is to determine a rule that describes the **rate** at which the height of the liquid is changing. The data collected in Question one is a record of the height at some time  $t$ . A visual inspection of this data combined with knowledge regarding the shape of the vase can produce answers to questions such as: "When is the height of the liquid changing most rapidly?" It is however possible to determine a rule for this rate.

Consider the following:

$\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$	$\frac{dv}{dt} = \text{constant}$	This can be determined from information provided and collected already.
	$\frac{dh}{dv}$	This can be determined as the reciprocal of $\frac{dv}{dh}$

**Question: 8.**

Determine a rule for:  $\frac{dh}{dt}$  and hence where the height is increasing at a maximum rate. **Note:** The rate can be in terms of the height of the liquid  $h$ .

Using time taken to fill the 1 litre vase is 90 seconds (from video)...

$$\frac{dv}{dt} = \frac{1000}{90} = \frac{100}{9} \text{ cm}^3 \text{ s}^{-1}$$

$$\frac{dv}{dh} = 8.918h^6 - 0.005925h^5 + 0.1490h^4 - 1.623h^3 + 5.313h^2 + 15.77h + 8.676$$

$$\frac{dh}{dt} = \frac{100}{9(8.918h^6 - 0.005925h^5 + 0.1490h^4 - 1.623h^3 + 5.313h^2 + 15.77h + 8.676)}$$

$$\frac{d^2h}{dt^2} = 0 \quad \text{when } h = -0.984 \text{ or } h = 5.77 \text{ or } h = 16.4$$

Since  $h > 0$ ,  $h = -0.984$  is not a solution. When  $h = 5.77$  the height of the liquid is changing at the slowest rate (can use vase to support) so when  $h = 16.4$  the height of the liquid is changing at the maximum rate... which aligns very well with the narrowest part of the vase.

A function relating the height of the liquid and time can be determined by considering the following:

$$\frac{dv}{dt} = \text{constant} \quad \left\| \quad v(t) = \int \text{constant } dt \quad \begin{array}{l} \text{This can be determined from information provided and} \\ \text{collected already.} \end{array} \right.$$

In Question 6 a rule was established for the volume in terms of the height. The information above can be used to determine a relationship between volume and time.

**Question: 9.**

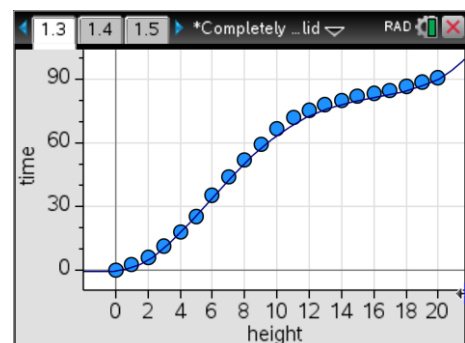
Write a rule relating *height* and *time* and define this rule as  $t(h)$  and discuss how well the function models the data. (See below)

$$\text{Since } \frac{dv}{dt} = \text{constant} \text{ and } v = 0 \text{ when } t = 0, \text{ then } v = \int \frac{1000}{90} dt = \frac{100t}{9}$$

$$\text{From Q 6: } v = 1.274 \times 10^{-5} h \cdot (h^6 - 77.51h^5 + 2338h^4 - 31855h^3 + 139017h^2 + 618705h + 680998)$$

$$\text{Then: } t(h) = 1.147 \times 10^{-6} h \cdot (h^6 - 77.51h^5 + 2338h^4 - 31855h^3 + 139017h^2 + 618705h + 680998)$$

The graph of  $t(h)$  is very close to all the data points. Small deviations can be attributed to measurement errors from the video and the approximation of the curve to the vase... all things considered ... this is a very good match.



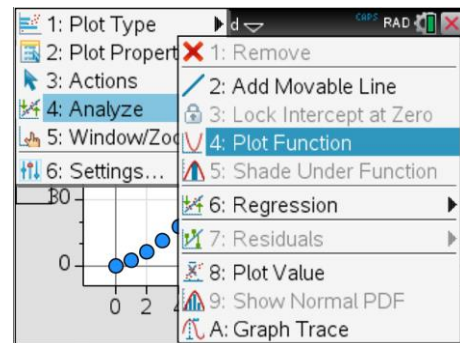
The rule defines the time in terms of the variable height, do not try and transpose the equation to obtain height in terms of time. The function can be validated using the instructions below.

Navigate back to page 1.3 and swap the axis... placing height on the horizontal (independent) axis and time on the vertical axis.

From the **Analyze** menu select **Plot Function**.

For the function enter:  $t(x)$

**Note:** The calculator plots functions in terms of  $x$ .



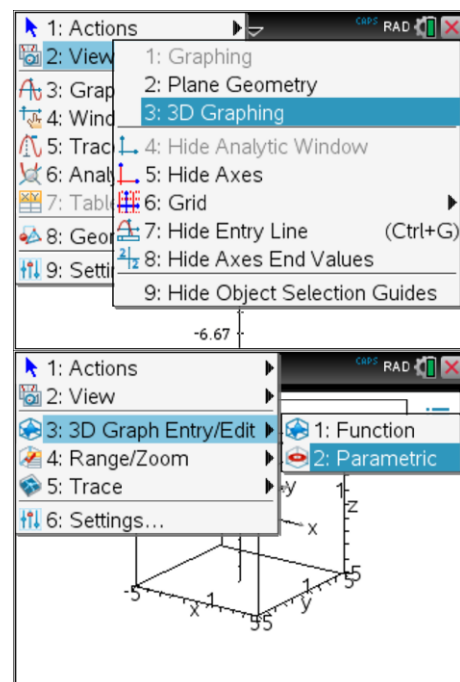
## Extension – Drawing the 3D Vase

Teacher notes: This section of the investigation does not require students to complete any 'questions' and is completed for 'interest' sake. Most students however like the 3D model. At least one school has used their 3D printer to produce a replica of a solid of revolution... now that's cool!

With equations relating to the vase validated, it is now possible to reproduce a 3D model of the vase on the calculator, a model that can be rotated and observed from any angle!

Insert a **Graph** application into the document.

Change the **View** to **3D Graphing**.



Change the **3D Graph Entry** option to **Parametric**.

To get a better view of the vase it is also worth hiding the 'box' that is displayed by default. To hide the box, use the **View** menu.

To view the entire vase the window settings need to be changed, there will be three sets of settings... as you will be viewing the vase in 3D.

From the **Range/Zoom** menu select **Range Settings**. The following settings are important:

$$X_{\min} = -8 \qquad X_{\max} = 8$$

$$Y_{\min} = -8 \qquad Y_{\max} = 8$$

$$Z_{\min} = -1 \qquad Z_{\max} = 21$$

Notice also in this menu, short cuts for zoom in / out are available by pressing  $\times$  to zoom in and  $\div$  to zoom out.

From the 3D graph screen, press [TAB] to access the equation entry lines. Three equations are entered and two parameters are used.

The rule that describes the shape of the vase is entered in both the x and y direction. This is achieved by using:  $f(t)$  to recall the rule for the shape of the vase defined in Question 3.

The expressions  $\sin(u)$  and  $\cos(u)$  rotate the curve around the axis and the  $t$  in  $zp1$  extends the graph along the z axis.

To see the entire vase the domain for the parameters  $u$  and  $t$  must be set. The parameter dialog box can be accessed by using the touchpad or by press [TAB] from the equation entry followed by [ENTER].

Adjust the settings as shown opposite. The value of  $t$  represents the minimum and maximum values for the height. The values for  $u$  represent the rotation achieved by the two functions  $\sin(u)$  and  $\cos(u)$ , therefore  $U_{\max} = 2\pi$ .

Your vase is ready!

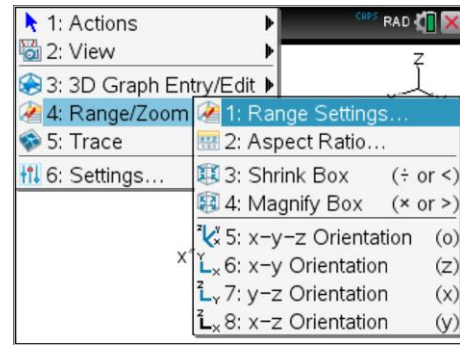
To see how the original equation is rotated around the axis to form this solid of revolution, edit the equations as follows:

$$xp1 \quad f(t) \cdot \sin(n \cdot u)$$

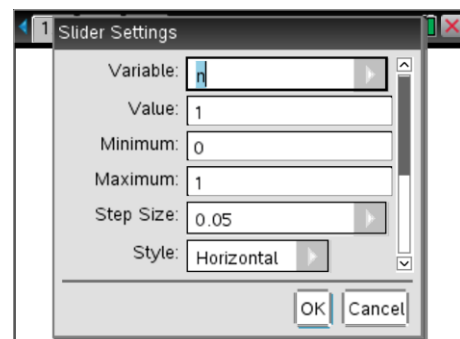
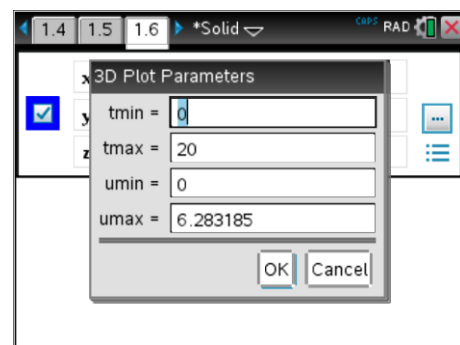
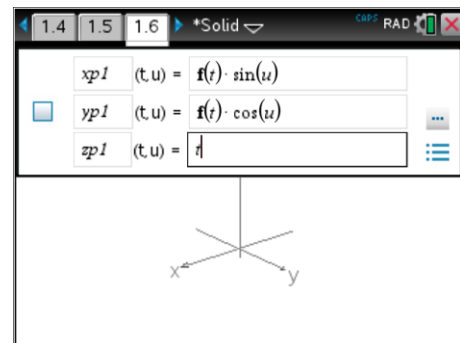
$$yp1 \quad f(t) \cdot \cos(n \cdot u)$$

$$zp1 \quad t$$

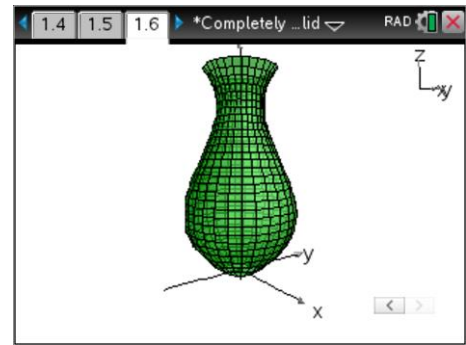
Use the **Actions** menu Insert a **Slider** (settings shown opposite), select okay and then vary the slider.



**Note:** Zoom in/out are different than the windows settings... try it once the vase is on the screen.



The completed 3D vase... students can change the view, remove the wire effect, change colour... and most importantly, rotate the graph (vase) so that it is visible from all angles.



Animate the slider to watch the jug form, the visual helps students in their understanding of solids of revolution.

