

# The Fibonacci Sequence

## Student Activity

7 8 9 10 11 12



## Introduction

There are an enormous number of amazing sequences and further patterns within each sequence. This investigation looks specifically at the Fibonacci sequence, one of the most famous of all sequences. The sequence is named after Leonardo Pisano (Leonardo of Pisa), his original name was Leonardo Fibonacci.

## Teacher Notes

Prior to commencing this investigation, students should be encouraged to watch the video tutorials on Sequences and Series on the Texas Instruments Australia website. These tutorials will help students with the calculator skills that will make this exploration more fluent.

## Fibonacci

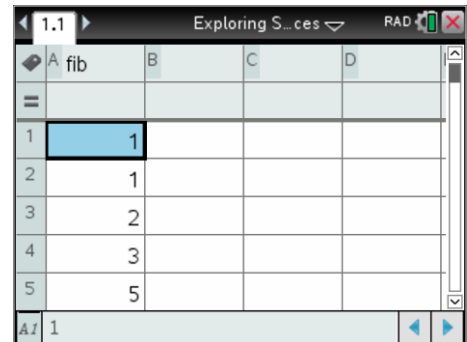
This is one of the most famous sequences: 1, 1, 2, 3, 5, 8, 13 ...

Each number is the sum of the previous two numbers. In its raw form the first two numbers are 1 and 1.

Use either a recursive formula or an appropriate sequence command to generate the first 100 terms of the Fibonacci sequence.

Call the list: FIB

Insert a Calculator Application in preparation for your exploration.



A	B	C	D
1	1		
2	1		
3	2		
4	3		
5	5		

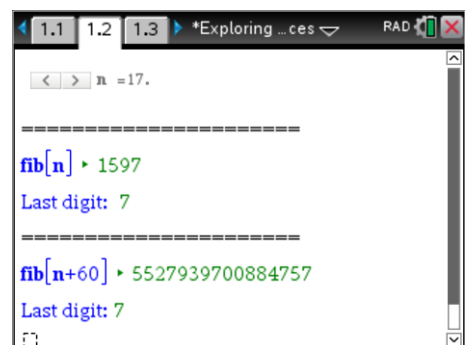
### Question: 1.

Explore the **last** digit of each Fibonacci number from the 61<sup>st</sup> term onwards. What do you notice?

Note: Any term in the sequence can be recalled by typing: `Fib[ # ]` where # represents the term number.

**Answer:** As an exploration, student answers will vary, however students should identify that the least significant digit of the numbers from the 60<sup>th</sup> Fibonacci number are the same as the first part of the sequence.

Students should be encouraged to consider the natural implication that these digits will continue to repeat. The image shown here is from a Notes application combined with an appropriate formula to extract the least significant digit.



```
1.1 1.2 1.3 *Exploring ...ces RAD
< > n =17.
=====
fib[n] ▶ 1597
Last digit: 7
=====
fib[n+60] ▶ 5527939700884757
Last digit: 7
```

**Calculator Tip!**

Insert a Notes application and a slider called 'n'. Set the minimum value of the slider to 1, the maximum to 40 with increments of '1'. In two separate and consecutive maths boxes type:

$\text{Fib}[n]$ ,

$\text{Fib}[n+60]$

**Question: 2.**

What happens when you square consecutive Fibonacci numbers and add them together? Try this for the first couple of Fibonacci numbers and write down the result. Where have you seen this set of numbers before?

Example:  $1^2 + 1^2 = 2$

$$1^2 + 2^2 = 5$$

$$2^2 + 3^2 = 13$$

**Answer:** Once again, as an exploration, student answers will vary, however students should identify that these squared terms generate every second term of the Fibonacci sequence.

Students could use the Spreadsheet application or a Note application to explore this pattern. The Notes application draws focus to each expression while the Spreadsheet will help identify the overall pattern. (Spreadsheet example shown here) Notice the simplicity with which all the squared terms can be calculated followed by a formula in cell C1: " $=B1 + B2$ " which is then filled down to populate the spreadsheet.

	A fib	B	C	D
=		=fib^2		
1	1	1	2	
2	1	1	5	
3	2	4	13	
4	3	9	34	
5	5	25	89	

**Question: 3.**

Square each of the numbers in the Fibonacci sequence. Find the progressive sum of these squared values. Compare this result to the product of consecutive Fibonacci numbers. What do you notice?

Example:  $1^2 + 1^2 = 2$

$$1 \times 1 = 1$$

$$1^2 + 1^2 + 2^2 = 6$$

$$1 \times 2 = 2$$

$$1^2 + 1^2 + 2^2 + 3^2 = 15$$

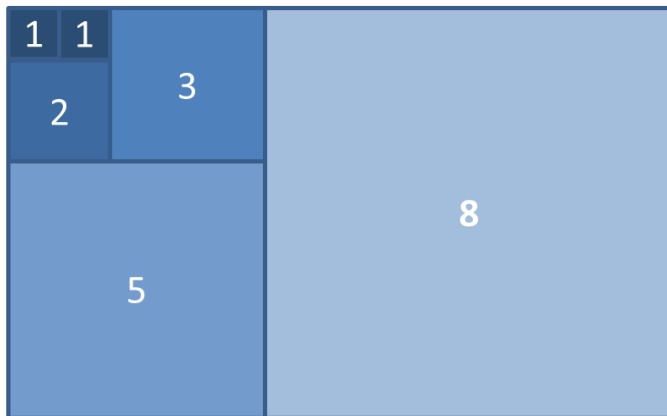
$$2 \times 3 = 6$$

**Answer:** This pattern is based on the geometrical interpretation of the construction of the Fibonacci sequence (Refer Question 4). It is another interesting pattern within the Fibonacci sequence.

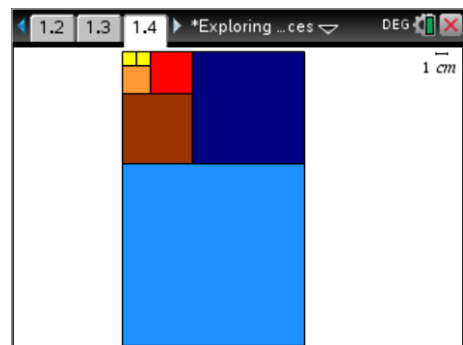
	A fib	B	C	D
=		=fib^2		
1	1	1	1	
2	1	1	2	
3	2	4	6	
4	3	9	15	
5	5	25	40	

**Question: 4.**

Study the diagram below. Explain how it is constructed, how it is related to the Fibonacci sequence and how it illustrates the pattern observed in Question 3.



**Answer:** Equally as important to finding the patterns is the 'why'. This geometric construction shows that side lengths are effectively the sum of the two most recent terms within the sequence. The total area can therefore be computed using this product. The other way to determine the area of this shape is to square all the previous terms in the sequence and add them together. This is therefore the same as the numerical approach to the problem.

**Question: 5.**


The ratio between consecutive Fibonacci terms is interesting and has its own name.

Explore this ratio as  $n \rightarrow \infty$  for  $\frac{fib[n+1]}{fib[n]}$ .

**Calculator Tip!**

Insert another Notes application and a slider called 't'. Set the minimum value of the slider to 1, the maximum to 100 with increments of '1'. In a maths boxes type:

$$\text{Approx}( \text{Fib} [t+1] / \text{Fib} [ t ] )$$

Once you believe you know enough about the value store it as  $\phi$  on the calculator. The  $\phi$  symbol is located in the symbols palette ( ctrl +  ), once stored the value can be retrieved from the VAR menu.

a) Square the number  $\phi$ . What do you notice?

**Answer:**  $\phi \approx 1.6180339887$   
 $\phi^2 = 1 + \phi$  When the ratio is squared the result is the same as adding 1 to the ratio.

- b) Use the result from part(a) to write an algebraic expression, solve the expression and hence determine the exact value of  $\phi$ .

Answer:

$$\begin{aligned}\phi^2 &= 1 + \phi \\ \phi^2 - \phi - 1 &= 0 \\ \left(\phi - \frac{1}{2}\right)^2 - \frac{1}{4} - 1 &= 0 \\ \left(\phi - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(\phi - \frac{1}{2} + \frac{\sqrt{5}}{2}\right) &= 0 \\ \phi = \frac{1 + \sqrt{5}}{2} \quad \text{or} \quad \phi = \frac{1 - \sqrt{5}}{2} \\ \text{Since } \phi > 0 \text{ then } \phi &= \frac{1 + \sqrt{5}}{2}\end{aligned}$$

- c) Calculate the reciprocal of  $\phi$ , what do you notice?

Answer: Students may attempt this question numerically on the calculator or by using the exact value from the previous question.

$$\begin{aligned}\frac{1}{\phi} &= \frac{2}{1 + \sqrt{5}} \\ \frac{1}{\phi} &= \frac{2}{1 + \sqrt{5}} \times \frac{1 - \sqrt{5}}{1 - \sqrt{5}} \\ \frac{1}{\phi} &= \frac{2 - 2\sqrt{5}}{1 - 5} \\ \frac{1}{\phi} &= \frac{\sqrt{5} - 1}{2} \\ \frac{1}{\phi} &= \frac{1 + \sqrt{5}}{2} - \frac{2}{2} \\ \frac{1}{\phi} &= \phi - 1\end{aligned}$$

d) Test this quantity:  $\phi^1 + \phi^2 = \phi^3$ , does it work for any other powers?

Answer: Students may attempt this question numerically on the calculator or by using the exact value from a previous question. This relationship holds for  $\phi^n + \phi^{n+1} = \phi^{n+2}$

$$\begin{aligned}\phi + \phi^2 &= \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^2 & \phi^3 &= \left(\frac{1+\sqrt{5}}{2}\right)^3 \\ &= \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1+2\sqrt{5}+5}{4}\right) & &= \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \\ &= \left(\frac{2+2\sqrt{5}}{4}\right) + \left(\frac{6+2\sqrt{5}}{4}\right) & &= \left(\frac{6+2\sqrt{5}}{4}\right)\left(\frac{1+\sqrt{5}}{2}\right) \\ &= \frac{8+4\sqrt{5}}{4} & &= \frac{6+6\sqrt{5}+2\sqrt{5}+10}{8} \\ &= 2+\sqrt{5} & &= \frac{16+8\sqrt{5}}{8} \\ & & &= 2+\sqrt{5}\end{aligned}$$

Note that it could also be concluded that since  $\phi^2 = 1 + \phi$  (Part B) then  $\phi^3 = \phi(1 + \phi) = \phi + \phi^2$

#### Teacher Notes:

This final problem could also be extended to a proof by induction.

The Fibonacci sequence also appears in the 'shallow diagonals' of Pascal's triangle! If the numerical patterns within the Fibonacci sequence have not convinced you about the amazing patterns and beauty, try searching for images related to the Fibonacci sequence on the World Wide Web. Nature is abundant with examples of this amazing sequence.