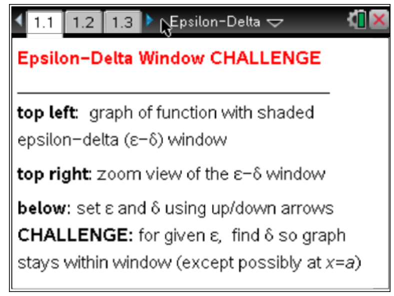
Epsilon-Delta Window Challenge
Student Activity

Name ________________
Class ________________

Open the TI-Nspire document *Epsilon-Delta.tns.*

In this activity, you will get to visualize what the formal definition of *limit* means in terms of a graphing window challenge.



Informally, $\lim_{x\to a} \mathbf{f}(x) = L$ means that the values of the output $\mathbf{f}(x)$ approach (get close to) L as the values of the input x approach a. This language is not very precise mathematically. Here is the formal mathematical definition of limit: $\lim_{x\to a} f(x) = L$ means:

> for any positive number $\varepsilon > 0$ (ε is the Greek letter "epsilon") there exists a positive number $\delta > 0$ (δ is the Greek letter "delta") such that $L - \varepsilon < f(x) < L + \varepsilon$ whenever $x \neq a$ and $a - \delta < x < a + \delta$.

The definition is heavy on symbols, but it provides the precision you need to make mathematical sense out of words like "get close to." The inequality $L - \varepsilon < f(x) < L + \varepsilon$ means the difference between $\mathbf{f}(x)$ and L is less than ε, while the inequality $a - \delta < x < a + \delta$ means that the difference between x and a is less than δ. When you write $\lim_{x\to a} f(x) = L$, it means that you are claiming that no matter how small a positive number ε you are given, you can choose another positive number δ so that you can guarantee the output $\mathbf{f}(x)$ is always between the two numbers $L - \varepsilon$ and $L + \varepsilon$ as long as $x \neq a$ and the input x is between $a - \delta$ and $a + \delta$.

You can make visual sense of it by thinking in terms of a graphing window challenge:
Consider the graph of $y = \mathbf{f}(x)$ in a window determined by two alternating opposing players.

- Your opponent always goes first and is allowed to set YMin (bottom) and YMax (top) values for the window, making sure the horizontal line $y = L$ is centered halfway between. To do that, your opponent must choose a positive number ε and set YMin = $L - \varepsilon$ and YMax = $L + \varepsilon$.

- Now it is your turn. You get to set XMin (left edge for the window) and XMax (right edge for the window) so that the vertical line $x = a$ is centered down the middle. That means you must pick a positive number δ with XMin = $a - \delta$ and XMax = $a + \delta$. Your goal is to guarantee that the graph of $y = \mathbf{f}(x)$ enters from the left side of the window and exits to the right, without ever leaving through the bottom or the top of the window, except possibly at the one value $x = a$. In other words, you must make sure that the value of $y = \mathbf{f}(x)$ is between the horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ for all values of x between $x = a - \delta$ and $x = a + \delta$. (Note: The value of $\mathbf{f}(a)$ does not matter at all for this game and could even be undefined.) If you succeed, then your choice of δ wins the challenge posed by your opponent's choice of ε. If you can always win the challenge, no matter what positive ε your opponent chooses, then you can say that $\lim_{x\to a} f(x) = L$.

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Press ctrl ▶ and ctrl ◀ to navigate through the lesson.

1. If **f1**(x) = x^2, a = 2, and L = 4, then you are interested in the question: is $\lim_{x \to 2} \mathbf{f1}(x) = 4$?

   a. At what point should the epsilon-delta (ε-δ) window be centered? _____________

   b. If your opponent sets ε = 1 and you set δ = 0.5, what will be the dimensions (XMin, XMax, YMin, YMax) for the epsilon-delta window?
   XMin = _________ XMax = _________ YMin = _________ YMax = _________

Move to page 1.3.

   c. Your opponent chose ε = 1, so set ε = 1 using the up/down arrows on the left. Now use the up/down arrows on the right to set δ = 0.5. Does this value δ = 0.5 meet the challenge? Why or why not? If not, can you find a value δ that does meet the challenge for ε = 1?

   d. Suppose your opponent chooses ε = 0.1. Set this value for ε and use the up/down arrows on the right to find a value for δ that meets the challenge. δ = _________

   e. Would a value of δ larger than the value you chose meet the ε = 0.1 challenge? Explain your answer.

   f. Would a value of δ smaller than the value you chose meet the ε = 0.1 challenge? Explain your answer.

   g. Suppose your opponent chooses ε = 0.01. Set this value for ε and use the up/down arrows on the right to find a value for δ that meets the challenge. δ = _________

h. Do you believe that you could always win the challenge for any given $\varepsilon > 0$? Why or why not?

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2. The function **f1** has a split definition, with the formula $\mathbf{f1}(x) = \begin{cases} 0.5x^2, & x \le 2 \\ 0.5x + 1, & x > 2 \end{cases}$. You are interested in the question: is $\lim_{x \to 2} \mathbf{f1}(x) = 2$?

Move to page 2.2.

a. Suppose your opponent chooses $\varepsilon = 1$. Set $\varepsilon = 1$ using the up/down arrows on the left. Now use the up/down arrows on the right to set $\delta = 0.5$. Does the value $\delta = 0.5$ meet the challenge? How do you know?

b. Suppose your opponent chooses $\varepsilon = 0.001$. Set this value for ε and use the up/down arrows on the right to find a value for δ that meets the challenge. $\delta =$ _________

c. Would a value of δ larger than the value you chose meet the $\varepsilon = 0.001$ challenge? Explain your answer.

d. Would a value of δ smaller than the value you chose meet the $\varepsilon = 0.001$ challenge? Explain your answer.

e. Do you believe that you could always win the challenge for any given $\varepsilon > 0$? Why or why not?

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Move to page 3.1.

3. The function **f1** here also has a split definition, with $\mathbf{f1}(x) = \begin{cases} 0.5x^2, & \text{if } x < 2 \\ 0.5x + 1.5, & \text{if } x \geq 2 \end{cases}$. Is $\lim_{x \to 2} \mathbf{f1}(x) = 2$?

Move to page 3.2.

a. Suppose your opponent chooses $\varepsilon = 1$. Does the value $\delta = 0.5$ meet the challenge? How do you know?

b. Suppose your opponent chooses $\varepsilon = 0.1$. Can you meet this challenge? Why or why not?

c. Explain why $\lim_{x \to 2} \mathbf{f1}(x) \neq 2$. Is there another value L such that $\lim_{x \to 2} \mathbf{f1}(x) = L$ for this function?

Move to page 4.1.

4. The function **f1** is not defined at $x = 1$.

a. Why is **f1**(1) undefined? Does this mean $\lim_{x \to 1} \mathbf{f1}(x)$ cannot exist?

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b. Investigate whether or not $\lim_{x \to 1} \mathbf{f1}(x) = 3$. Try meeting the ε challenge for $\varepsilon = 0.1$, 0.01, and 0.001 by finding an appropriate δ for each.

$\varepsilon = 0.1$, $\delta =$ _______ $\varepsilon = 0.01$, $\delta =$ _______ $\varepsilon = 0.001$, $\delta =$ _______

c. Is it possible to meet the ε challenge for any positive ε by finding an appropriate δ?

Extension

Move to page 5.1.

5. On page 5.1, the function **f1** has a split definition. On page 5.2, you will see that the graph looks very much like the function in problem 2.

   a. Is $\lim_{x\to 2} \mathbf{f1}(x) = 2$?

Move to page 5.2.

   b. Suppose you get to go first and choose ε. Is there a positive value ε such that there is no δ that meets the challenge?

   c. What does your answer to part 5b say about whether $\lim_{x\to 2} \mathbf{f1}(x) = 2$?

Move to page 6.1.

6. Examine the function given on page 6.1.

   a. Is $\lim_{x\to 1} \mathbf{f1}(x) = 2$? Why or why not?

Move to page 6.2.

   b. Suppose your opponent sets $\varepsilon = 0.00000000001$. What is the *largest* δ you could use to meet the challenge?

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Move to page 7.1.

7. Let $\mathbf{f1}(x) = \sin\left(\frac{1}{x}\right)$.

   a. This function is undefined at $x = 0$. Why?

Move to page 7.2.

   b. Is $\lim_{x \to 0} \mathbf{f1}(x) = 0$? Defend your answer.

Move to page 8.1.

8. $\mathbf{f1}(x) = x \cdot \sin\left(\frac{1}{x}\right)$

   a. This function is undefined at $x = 0$. Why?

Move to page 8.2.

   b. Is $\lim_{x \to 0} \mathbf{f1}(x) = 0$? Defend your answer.