String Graphs – Part 2

Answers

7 8 9 10 **11** 12

TI-Nspire Investigation Student 45 min

Aim

- Determine the parametric equation for the curve created by the successive intersection points of lines passing through points on $y = x$ and $y = -x$

Determining Equations

Start a **new document** and insert a **Graph application**.

Use the **[Menu]** to adjust the window settings:

Xmin = -10

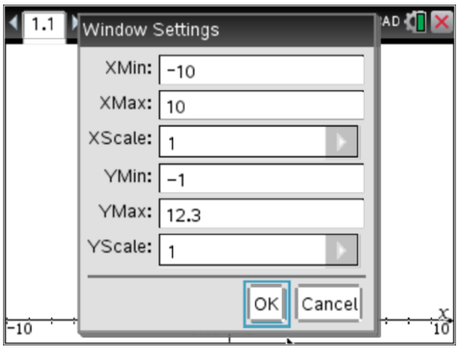
Xmax = 10

Ymin = -1

Ymax = 12.3

Graph the lines $y = x$ and $y = -x$.

It is worthwhile changing the attributes of these lines to: *dotted.*

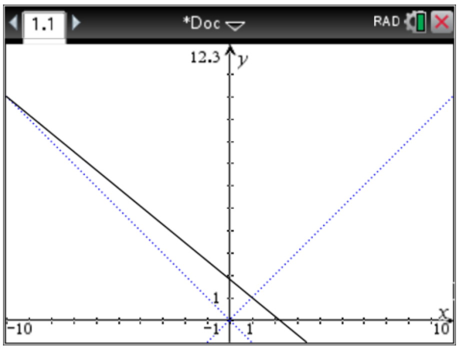


A series of straight line graphs will be constructed to form a string pattern.

The first straight line graph passes through the points:

(-10, 10) & (1, 1)

The result is shown opposite. Use the questions to help determine the equation for this line and all subsequent lines.



The equation to any straight line can be expressed in the form: $y = mx + c$

In this task it may be useful to express the equations in the form: $y = m(x - h) + k$

m = gradient = $\frac{rise}{run}$

(h, k) = represents the coordinates of a point that the line passes through

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Question: 1.

Determine the equation of this first line, passing through the points: (-10, 10) & (1, 1)

a) Calculate the gradient of the first line. $m=\frac{-9}{11}$

b) Nominate a point for the line to pass through and hence write down the equation in the form:

$y=m(x-h)+k$. Answers may vary: Selected Point (1, 1) Equation $y=\frac{-9}{11}(x-1)+1$

c) Write down the equation in the form: $y=mx+c$. $y=\frac{-9}{11}x+\frac{20}{11}$

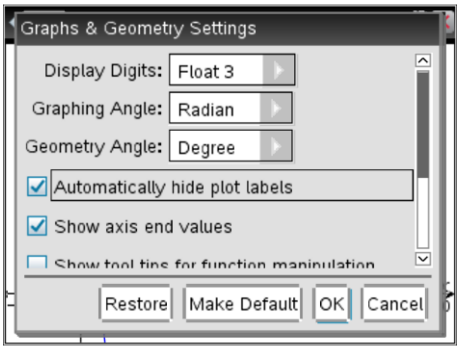
Once the first line is completed, try the second line.

The second straight line graph passes through the points:

(-9, 9) & (2, 2)

As more graphs are added it may be desirable to remove the equation labels.

Settings > Automatically hide plot labels



Question: 2.

Determine the equation of the line, passing through the points: (-9, 9) & (2, 2).

Answers may vary depending on equation format and selected point. $y=\frac{-7}{11}(x-2)+2$

Question: 3.

Determine the gradient and y – intercept for the remaining straight lines in this family of lines. Record your results using exact values. Graph all 10 equations on the same set of axis.

Eqn. No.	Point 1	Point 2	Gradient	Equation
1	(-10, 10)	(1, 1)	$\frac{-9}{11}$	$y=\frac{-9}{11}(x-1)+1$ or $y=\frac{-9}{11}x+\frac{20}{11}$
2	(-9, 9)	(2, 2)	$\frac{-7}{11}$	$y=\frac{-7}{11}(x-2)+2$ or $y=\frac{-7}{11}x+\frac{36}{11}$
3	(-8, 8)	(3, 3)	$\frac{-5}{11}$	$y=\frac{-5}{11}(x-3)+3$ or $y=\frac{-5}{11}x+\frac{48}{11}$
4	(-7, 7)	(4, 4)	$\frac{-3}{11}$	$y=\frac{-3}{11}(x-4)+4$ or $y=\frac{-3}{11}x+\frac{56}{11}$
5	(-6, 6)	(5, 5)	$\frac{-1}{11}$	$y=\frac{-1}{11}(x-5)+5$ or $y=\frac{-1}{11}x+\frac{60}{11}$

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6	(-5, 5)	(6, 6)	$\frac{1}{11}$	$y=\frac{1}{11}(x-6)+6$ or $y=\frac{1}{11}x+\frac{60}{11}$
7	(-4, 4)	(7, 7)	$\frac{3}{11}$	$y=\frac{3}{11}(x-7)+7$ or $y=\frac{3}{11}x+\frac{56}{11}$
8	(-3, 3)	(8, 8)	$\frac{5}{11}$	$y=\frac{5}{11}(x-8)+8$ or $y=\frac{5}{11}x+\frac{48}{11}$
9	(-2, 2)	(9, 9)	$\frac{7}{11}$	$y=\frac{7}{11}(x-9)+9$ or $y=\frac{7}{11}x+\frac{36}{11}$
10	(-1, 1)	(10, 10)	$\frac{9}{11}$	$y=\frac{9}{11}(x-10)+10$ or $y=\frac{9}{11}x+\frac{20}{11}$

A single equation can be determined to graph all 10 equations by using a parameter (t) for the equation number. Study each of your equations above and compare with the equation 'number'.

Question: 4.

Write the general equation in the form: $y=\frac{a}{b}(x-h)+k$ where a, b, h and k are expressions in terms of t.

a) Determine an expression for a in terms of t. $a=2t-11$

b) Determine an expression for b in terms of t. $b=11$

c) Determine an expression for h and k in terms of t. $h=t$ and $k=t$

d) Write down the general equation for the family of straight lines in the form: $y=\frac{a}{b}(x-h)+k$.

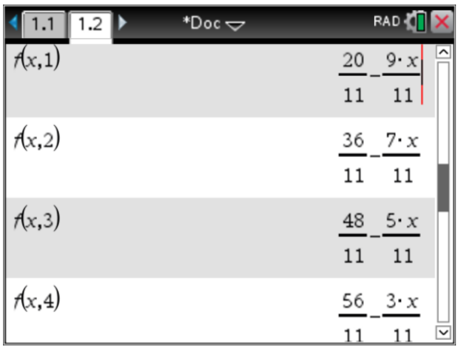
$$y=\frac{2t-11}{11}(x-t)+t$$

Note: Some students may use points on the line $y=-x$ however these will simplify to the same.

e) Write down the general equation for the family of straight lines in the form: $y=mx+c$.

$$y=\frac{2t-11}{11}x+\frac{2t(11-t)}{11}$$

f) Verify your equations by substituting a range of values for t and comparing with the corresponding original equation.
Answers will vary.



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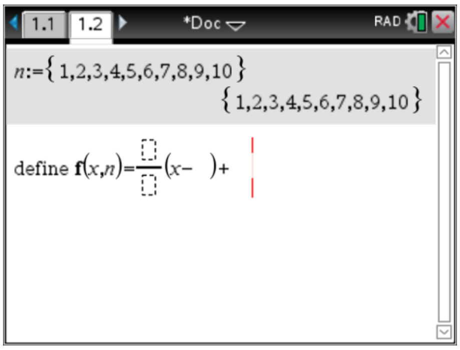
Insert a Calculator application and define *t* as the set of integers: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Define your general equation in terms of the variable *x* and parameter *t*.

Return to the Graph application and graph your function:

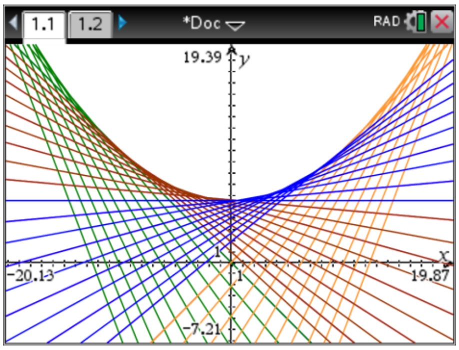
$$f(x,t)$$



Graph the following extended family of straight lines:

- $f(x,t)$,
- $f(x,t+10)$
- $f(x,t-10)$
- $f\left(x,\frac{t}{2}\right)$

To see the full effect, zoom out using the zoom out tool in the Window / Zoom menu and place the magnifying glass close to the centre of the screen.



Question: 5.

Describe the shape of the curve formed by the extended family of straight lines.

Curve appears to be a parabola.

Note: Students may enter the equation $y = x^2$ and then dilate and translate this function by dragging it around the calculator screen.

Finding a Locus

The points of intersection between successive equations can be used to produce the curve where infinitely many straight lines are generated.[1]

Question: 6.

Show that the first two lines passing through (-10, 10) & (1, 1) and (-9, 9) & (2, 2) intersect when:

$$x = -8 \text{ and } y = \frac{92}{11}.$$

Equations: $y = \frac{-9}{11}x + \frac{20}{11}$ and $y = \frac{-7}{11}x + \frac{36}{11}$

$$-9x + 20 = -7x + 36$$
$$-16 = 2x$$
$$x = -8$$

therefore: $y = \frac{-9}{11}(-8) + \frac{20}{11}$

$$y = \frac{92}{11}$$

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[1] The original curve or envelope would be tangent to the straight line equations, as the number of lines over the interval is increased successive points of intersection would come closer and closer to the curve.

Question: 7.

Use simultaneous equations to determine the next point of intersection, between equations 2 and 3.

Equations: $y=\frac{-7}{11}x+\frac{36}{11}$ and $y=\frac{-5}{11}x+\frac{48}{11}$

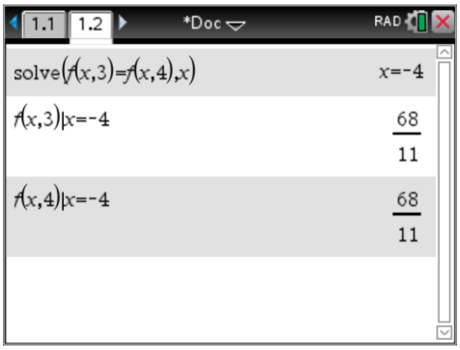
$$-7x+36=-5x+48$$
$$-12=2x$$
$$x=-6$$

therefore:

$$y=\frac{-7}{11}(-6)+\frac{36}{11}$$
$$y=\frac{78}{11}$$

Question: 8.

Use CAS to determine the point of intersection between $f(x,3)$ and $f(x,4)$.



Question: 9.

Complete the table below for the points of intersection between successive lines.

Question: 10.

Use the difference table to help identify the nature of the pattern in the y coordinates. Based on the results determine an equation in terms of the equation number t.
Note: When t = 1 this will be the point of intersection between equations 1 and 2. When t = 2, this will be the point of intersection between equations 2 and 3.

Eqn. Nos.	Point of Intersection
1 & 2	$\left(-8,\frac{92}{11}\right)$
2 & 3	$\left(-6,\frac{78}{11}\right)$
3 & 4	$\left(-4,\frac{68}{11}\right)$
4 & 5	$\left(-2,\frac{62}{11}\right)$
5 & 6	$\left(0,\frac{60}{11}\right)$

y-Coordinate	Δ_1	Δ_2
$\frac{92}{11}$	$\frac{-14}{11}$	$\frac{4}{11}$
$\frac{78}{11}$	$\frac{-10}{11}$	$\frac{4}{11}$
$\frac{68}{11}$	$\frac{-6}{11}$	$\frac{4}{11}$
$\frac{62}{11}$	$\frac{-2}{11}$	$\frac{4}{11}$
$\frac{60}{11}$	$\frac{2}{11}$	$\frac{4}{11}$

6 & 7	$\left(2, \frac{62}{11}\right)$
7 & 8	$\left(4, \frac{68}{11}\right)$
8 & 9	$\left(6, \frac{78}{11}\right)$
9 & 10	$\left(8, \frac{92}{11}\right)$

$\frac{62}{11}$	$\frac{6}{11}$	$\frac{4}{11}$
$\frac{68}{11}$	$\frac{10}{11}$	$\frac{4}{11}$
$\frac{78}{11}$	$\frac{14}{11}$	$\frac{4}{11}$
$\frac{92}{11}$		

The constant second order difference shows that the general equation for the y coordinates for the points of intersection is quadratic. There are numerous ways to produce such an equation either by hand using simultaneous equations or on CAS. If students are solving by hand, they should recognise the symmetry of the solutions and also note that when *t* = 5 the point of intersection is on the *y* axis. The equation must therefore be of the form:

$y = m(t-5)^2 + \frac{60}{11}$ By substitution of another point of intersection: $y = \frac{2}{11}(t-5)^2 + \frac{60}{11}$

Using the CAS it is possible to solve: $\text{solve}\left(f(x,t) = f(x,t+1), x\right)$ produces: $x = 2t - 10$.

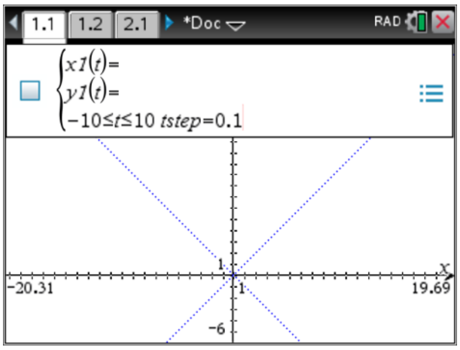
Therefore $f(x,t) \mid x = 2t-10$ results in: $y = \frac{2t^2}{11} - \frac{20t}{11} + 10$

Question: 11.

Write down the coordinates for successive points of intersection in terms of *t*.

$$\left(2t-10, \frac{2t^2}{11} - \frac{20t}{11} + 10\right)$$

On the Graph application, change the graph type to parametric and use the equations from Question 12 for the *x* coordinate and Question 13 for the *y* coordinate. Change the step size to 0.1 and the domain for *t*: -10 ≤ t ≤ 10.



Question: 12.

Write down the coordinates for successive points of intersection as a polynomial.

$$y = \frac{2t^2}{11} - 2t + 11 \quad \text{and} \quad x = 2t - 10$$

$$\text{Therefore} \quad t = \frac{x+10}{2}$$

$$y = \frac{x^2}{22} + \frac{60}{11}$$

Author: P. Fox

Extension 1 – Refining the Rule

Question: 13.

The curve created by the lines is sometimes referred to as an ‘envelopee’ and is slightly different than the curve that passes through successive points of intersection.

a. If twice as many lines were drawn, creating twice as many points of intersection, would the curve through the points of intersection be different?
Yes. The equation will be different as the points of intersection will be different. The equations however are very similar!

b. Compare each of the following:

- $\text{solve}\left(f(x,t)=f\left(x,t+\tfrac{1}{2}\right),x\right)$ $\quad x=2t-\dfrac{21}{2}$ $\quad x\approx 2t-10.5$
- $\text{solve}\left(f(x,t)=f\left(x,t+\tfrac{1}{10}\right),x\right)$ $\quad x=2t-\dfrac{109}{10}$ $\quad x\approx 2t-10.9$
- $\text{solve}\left(f(x,t)=f\left(x,t+\tfrac{1}{100}\right),x\right)$ $\quad x=2t-\dfrac{1099}{100}$ $\quad x\approx 2t-10.99$
- $\text{solve}\left(f(x,t)=f\left(x,t+\tfrac{1}{1000}\right),x\right)$ $\quad x=2t-\dfrac{10999}{1000}$ $\quad x\approx 2t-10.999$

c. Use the approach above to determine the equation for the curve to the ‘envelope’.

Students may draw the conclusion from the above that the equation for the ‘envelope’ is given by: $x=2t-11$. Alternative approaches include:

$\text{solve}\left(f(x,t)=f(x,t+a),x\right)$ produces $x=2t-11+a$ then $\lim\limits_{a\to 0}\left(2t-11+a\right)=2t-11$

$\text{solve}\left(f(x,t)=f\left(x,t+\tfrac{1}{a}\right),x\right)$ produces $x=\dfrac{2at-11a+1}{a}$ then $\lim\limits_{a\to 0}\left(\dfrac{2at-11a+1}{a}\right)=2t-11$

d. Determine the equation to the polynomial for the ‘envelope’.

$$y=\frac{2t^2}{11}-2t+11 \quad \text{and} \quad x=2t-11$$

$$y=\frac{x^2}{22}+\frac{11}{2}$$

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Extension 2 – Developing a General Rule

Question: 14.

The curve is now created by symmetrically stitching points on the lines $y = -mx$ and $y = mx$.

a. Determine the equation for the curve defined by the envelope formed as the lines: $y = -2x$ and $y = 2x$ are stitched together.

The points being stitched in this investigation include: (-10, 20); (-9, 18); (-8, 16) …

The general equation for the straight lines: $y = \frac{2(2t-11)}{11}x - \frac{4t(t-11)}{11}$

The equation for the envelope: $y = \frac{x^2}{11} + \frac{120}{11}$

For the limiting case where $\Delta t \to 0$ the equation for the envelope is: $y = \frac{x^2}{11} + \frac{121}{11}$

b. Determine the equation for the curve define by the envelope formed as the lines: $y = -3x$ and $y = 3x$ are stitched together.

The points being stitched in this investigation include: (-10, 30); (-9, 27); (-8, 16) …

The general equation for the straight lines: $y = \frac{3(2t-11)}{11}x - \frac{6t(t-11)}{11}$

The equation for the envelope: $y = \frac{3(x^2+120)}{22}$

For the limiting case where $\Delta t \to 0$ the equation for the envelope is: $y = \frac{3x^2}{22} + \frac{363}{22}$

c. Determine the equation for the curve defined by the envelope formed as the lines: $y = -mx$ and $y = mx$ are stitched together.

Determine the general polynomial equation for the curve defining the envelope.

The points being stitched in this investigation include: (-10, 10m); (-9, 9m); (-8, 8m) …

The general equation for the straight lines: $y = \frac{m(2t-11)}{11}x - \frac{2mt(t-11)}{11}$

The equation for the envelope: $y = \frac{m(x^2+120)}{22}$

For the limiting case where $\Delta t \to 0$ the equation for the envelope is: $y = \frac{m(x^2+121)}{22}$

Teacher Notes:

Students can use a slider to control the value of *m* and the entire family of straight lines will be regraphed every time the slider value is changed, this is a very dynamic and powerful way to see that the equations are correct.