String Graphs – Part 1 + 2

Answers

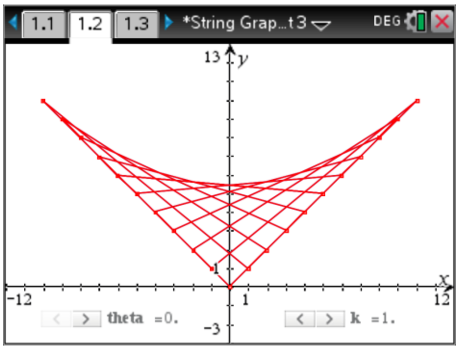
7 8 9 10 **11** 12

TI-Nspire Investigation Student 45 min

Aim

- Connect the outcomes of Advanced Strings Graphs Part 1 and Advanced Strings Part 2 using transformation matrices

Visualising the Connection



Open the TI-Nspire file String Graphs 3.

Page 1.2 contains a visual of the String Graphs produced in Activity 2. Two matrices control the location of these string patterns:

- Rotation matrix
- Dilation matrix

The angle θ (theta) is associated with the rotational matrix and can be changed using the slider. The dilation matrix dilates in both the x and y direction and can be adjusted using the k slider.

Adjust the sliders to map the lines and points from activity 2 to the lines and points from activity 1.

Question: 1.

What is the angle (measured in degrees) required to orient the points and lines from activity 2 back to those from activity 1? Justify your answer.

The line y = x makes a 45° angle with the x – axis. The points along the y = x line must be mapped back to the x axis, 45°. Similarly the line y = -x makes an angle of 45° with the y axis, the points along this line are mapped back to the y axis.

Question: 2.

What is the dilation factor required to map the points and lines from activity 2 back to those from activity 1? Justify your answer.

The approximate slider value that produces a reasonable mapping is: 0.7.

The point (10, 10) is $10\sqrt{2}$ units from the origin, this point is mapped to (10, 0) which is only 10 units from the origin. The points along the line y = x are equally spaced, so too those along the x-axis.

A dilation of $\frac{1}{\sqrt{2}}$ must therefore be applied to all points.

Similarly with the point (-10, 10) mapping to (0, 10) and all the points along the line y = -x.

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Question: 3.

Explain how the rotation and dilation connect the String Graphs activities 1 and 2.

The lines y = x and y = -x are perpendicular, so too the x and y axis.

The line y = x makes an angle of 45° with the x axis, so a rotation of 45° (clockwise) will map the points along the line y = x onto the x axis.

The point (1, 1) on the line y = x is $\sqrt{2}$ units from the origin; therefore a dilation must be included with the rotation to map the point (1, 1) to the point (1, 0).

- The dilation matrix $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ represents a dilation factor k from the y axis.
- The dilation matrix $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ represents a dilation factor k from the x axis.
- The rotational matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ produces a rotation of θ in a counter-clockwise direction about the origin.

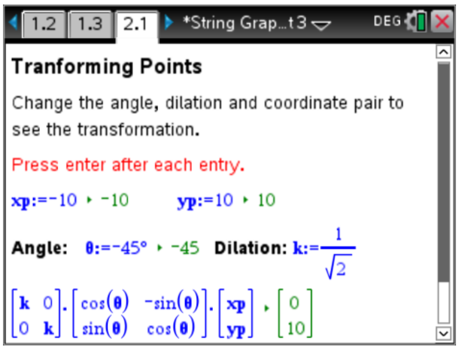
Teacher Notes:

To make use of the lists in the spreadsheet the coordinate pairs are set in rows rather than columns therefore requiring the rotation matrix to be transposed. Students may look at the formula structure and wonder why it appears differently than on the question sheet.

Navigate to problem 2, page 1.

In this Notes application the angle, dilation and coordinates can be edited (press **Enter** after each edit). The corresponding matrix entries will automatically update.

The image opposite shows one of the original points in Activity 2 (-10, 10) transformed via a rotation in a clockwise direction (-45°) and dilated by a factor of $\dfrac{1}{\sqrt{2}}$ from both the y and x axis.



The resulting coordinate (0, 10) corresponds to the first point in Activity 1.

In Activity 1 points on the y axis (0, 10), (0, 9) … were connected to points along the x axis (1, 0), (2, 0) ….
In Activity 2 points along the line $y=-x$, (-10, 10), (-9, 9) … were connected to points along the line $y=x$ (1, 1), (2, 2) …

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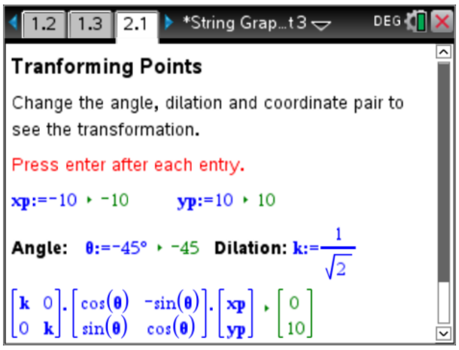
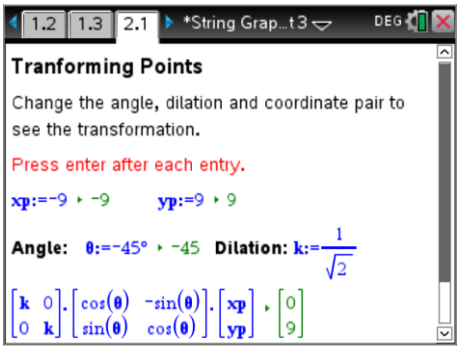
Question: 4.

Use the matrix transformations on Page 2.1 to show that the points in Activity 2 can be transformed to the original points in Activity 1.

Sample entries shown here ...

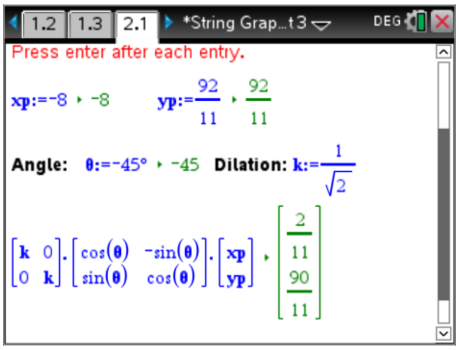




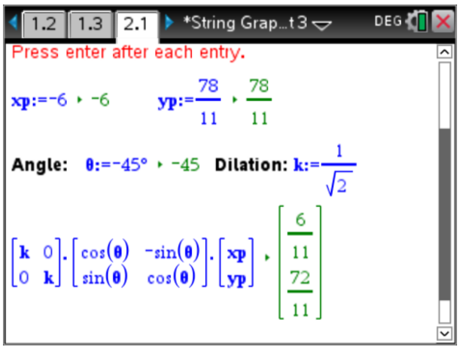
Question: 5.

The same matrix transformations on Page 2.1 can be applied to the points of intersection between consecutive lines. The first four points of intersection in Activity 2 are shown below. Determine their corresponding points in Activity 1 using the matrix transformations.

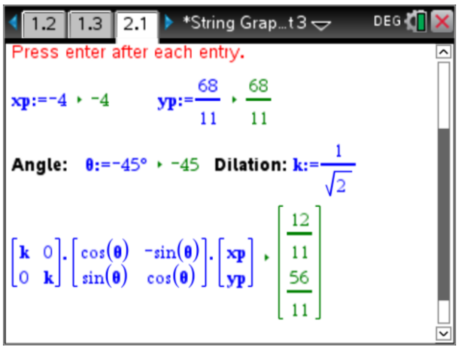
Point 1: $\left(-8, \frac{92}{11}\right)$



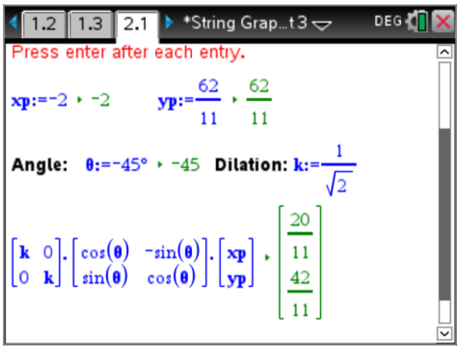
Point 2: $\left(-6, \frac{78}{11}\right)$



Point 3: $\left(-4, \frac{68}{11}\right)$



Point 4: $\left(-2, \frac{62}{11}\right)$



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Equations

The same transformations applied to the points from Activity 1 and 2 can be applied to the lines.

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x'\\ y'\end{bmatrix}$$

Expressions for x and y can be determined on the calculator using inverse matrix operations:

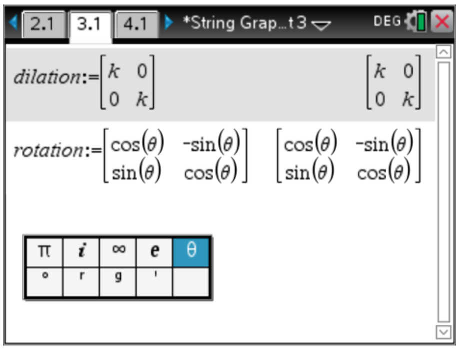
$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}^{-1}\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^{-1}\begin{bmatrix}x'\\ y'\end{bmatrix}$$

Navigate to page 3.1. The rotation and dilation matrices have already been entered so they can be copied and pasted as required.

Store -45° in angle θ

$\theta := -45^\circ$

Store $\frac{1}{\sqrt{2}}$ in the value for *k*.

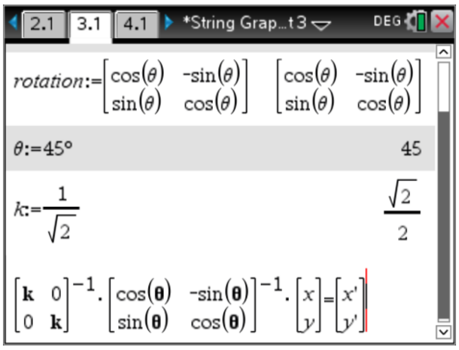


The matrix transformations can be entered naturally as they are expressed above.

Ctrl + C = Copy

Ctrl + V = Paste

An alternative method is to highlight the required expression and press [Enter] and the expression will be pasted into the active cursor position.



Question: 6.

Write expressions for x and y in terms of x' and y'.

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}}\end{bmatrix}^{-1}\begin{bmatrix}\cos(-45^\circ) & -\sin(-45^\circ)\\ \sin(-45^\circ) & \cos(-45^\circ)\end{bmatrix}^{-1}\begin{bmatrix}x'\\ y'\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x'-y'\\ y'+x'\end{bmatrix}$$

Question: 7.

The linear equation determined in Advanced String Graphs Part 2, passing through (-10, 10) and (1, 1) is given by: $y=-\frac{9}{11}x+\frac{20}{11}$. Use your result from Question 6 to determine the linear equation passing through the points (0, 10) and (1, 0) corresponding to the first equation in Advanced String Graphs Part 1.

Author: P. Fox

$x = x' - y' \quad \text{and} \quad y = y' + x'$

Use result from matrix equation.

Given: $y = -\frac{9}{11}x + \frac{20}{11}$

$y' + x' = -\frac{9}{11}(x' - y') + \frac{20}{11}$

Substitution of result into equation from Activity 2.

$y' - \frac{9}{11}y' = -\frac{9}{11}x' - x' + \frac{20}{11}$

Solve equation for y.

$2y' = -20x' + 20$

$y' = -10x' + 10$

Transformed Equation:

$y = -10x + 10$

Question: 8.

The linear equation determined in Advanced String Graphs Part 2, passing through (-9, 9) and (2, 2) is given by: $y = -\frac{7}{11}x + \frac{36}{11}$. Use your result from Question 6 to determine the linear equation passing through the points (0, 9) and (2, 0) corresponding to the first equation in Advanced String Graphs Part 1.

$x = x' - y' \quad \text{and} \quad y = y' + x'$

Use result from matrix equation.

Given: $y = -\frac{7}{11}x + \frac{36}{11}$

$y' + x' = -\frac{7}{11}(x' - y') + \frac{36}{11}$

Substitution of result into equation from Activity 2.

$y' - \frac{7}{11}y' = -\frac{7}{11}x' - x' + \frac{36}{11}$

Solve equation for y.

$4y' = -18x' + 36$

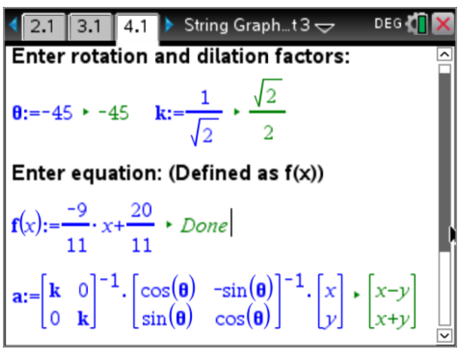
$y' = -\frac{9}{2}x' + 9$

Transformed Equation:

$y = -\frac{9}{2}x + 9$

Question: 9.

Navigate to page 4.1 and enter the appropriate transformations and equation, using the function notation provided: $f(x)$.



a) Check your answers to Questions 7 and 8.

Complete on calculator

b) Check the following two equations from Part 2.

Complete on calculator

Author: P. Fox

Question: 10.

The parabola passing through the points of intersection in Activity 2 was: $y = \frac{x^2}{22} + \frac{60}{11}$. Use an appropriate matrix transformation to write an equation for the equation to the curve from Activity 1.

Do not attempt to express the equation in the form y = . The equation can however be copied and pasted into the graph application on page 1.2 (Relation 1) to confirm it is correct.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x'-y' \\ y'+x' \end{bmatrix}$$

$$y + x = \frac{(x-y)^2}{22} + \frac{60}{11}$$

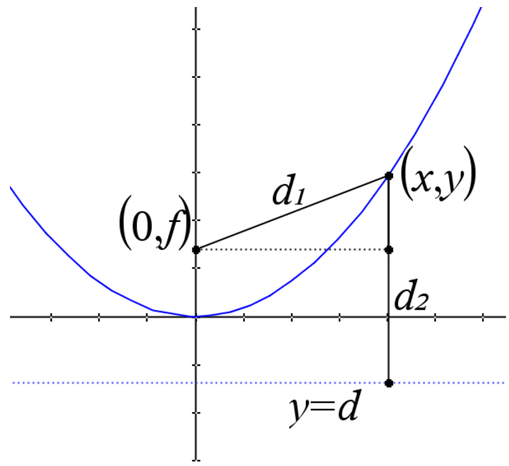
$$22(x+y) = x^2 - 2xy + y^2 + 120$$

$$x^2 - 22x + y^2 - 22y - 2xy + 120 = 0$$

Teacher Notes: The equation above can be graphed directly using the relational graphing option.

Extension – Conic Sections

A parabola is defined as a set of points equidistant from a single point (focus) and a line (directrix), that is: $d_1 = d_2$ in the diagram opposite.



Question: 11.

Use the equation from Activity 2: $y = \frac{x^2}{22} + \frac{60}{11}$ to determine the location of the focus and directrix.

$$\text{Solve}\left(\sqrt{x^2 + (y-f)^2} = y - d, y\right)$$

$$y = \frac{x^2}{2(f-d)} + \frac{f^2 - d^2}{2(f-d)}$$

Since: $y = \frac{x^2}{22} + \frac{60}{11}$, students can set up two equations and solve simultaneously.

Equations: $d - f = 11$ and $f^2 - d^2 = 60$

$$d = -\frac{1}{22} \quad \text{and} \quad f = \frac{241}{22}$$

Author: P. Fox

Teacher Notes: The Graph Menu: **Analyse Graphs** > **Analyse Conics** > **Foci** will provide the same result as above, similarly **Analyse Graphs** > **Analyse Conics** > **Directrix** will produce the other required result. Teachers may choose to include 'use the distance property to determine …" for this question.

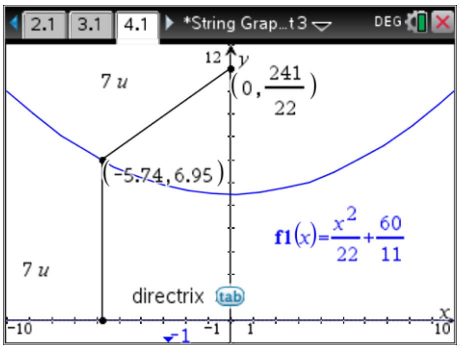
Question: 12.

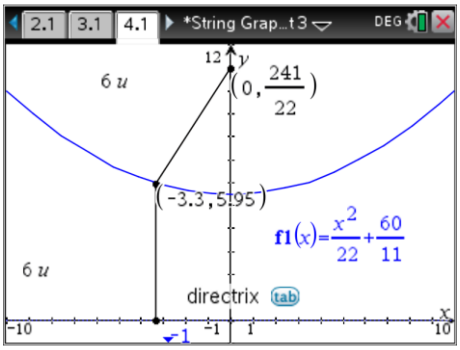
Check your answer to the previous question using a selection of points on the curve.

Answers will vary:

The simplest point to check first is the turning point: $\left(0, \dfrac{60}{11}\right)$ which is $\dfrac{11}{2}$ units from both the focus and directrix.

Students can also do an approximate check using the calculator by placing a point on the curve and measuring the distance from the focus to the point on the curve and also the point on the curve to the directrix.





Teacher Notes: Numerous paper folding and geometric constructions can also help students gain a better understanding of the construction of a parabola. A simple activity is to line students up along the front of the classroom. Another student (F) stands approximately 2 metres from the front of the room and equidistant from either *side* of the room. The students standing at the front of the room are then instructed to walk *forward* (perpendicular to the front of the room) and STOP when they believe they are the same distance from the front of the room as they are from student F. The front of the room represents the directrix and student F represents the focus. The *curve* that students are standing on is a parabola.

Question: 13.

Use transformation matrices to determine the coordinates of the focal point and equation to the directrix for the curve from Activity 1. Use a selection of points to show that your answer is correct.

Original Directrix Equation: $y=-\dfrac{1}{22}$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1}\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x'-y' \\ y'+x' \end{bmatrix}$$

$$y+x=-\frac{1}{22}$$

$$y=-x-\frac{1}{22}$$

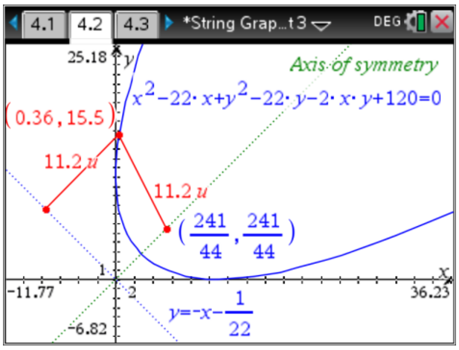
New Directrix Equation: $y=-x-\dfrac{1}{22}$

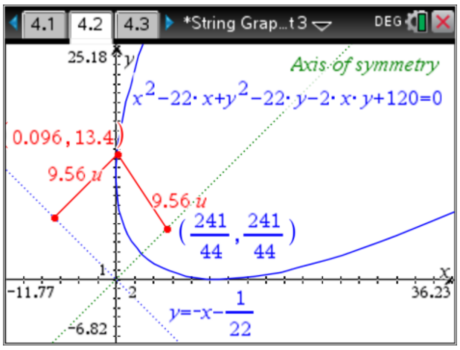
Original Focal Point: $\left(0,\dfrac{241}{22}\right)$

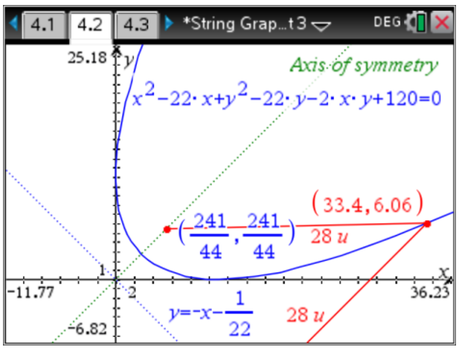
$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix}\begin{bmatrix} 0 \\ \frac{241}{22} \end{bmatrix}$$

AND $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \frac{241}{44} \\ \frac{241}{44} \end{bmatrix}$

Once again students can use the Graph Menu to analyse conics and check their answers. Placing points on the curve and measuring the distance from the focus to the point on the curve and then also from this point to the directrix provides a powerful visual for students to observe and verify that the equation they have graphed is still parabolic.







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