Triangular Numbers

Teacher Notes & Answers

7 8 9 10 11 **12**

TI-30XPlus MathPrint™ | Activity | Student | 50 min

Introduction to Induction

Teacher Notes:

Triangular numbers are so named because numbers when represented by dots can form a triangular pattern. The triangular numbers are 1, 3, 6, 10, 15 … This activity uses a range of visual and numerical approaches to generate a formula for triangular numbers, however this should not be considered as 'proof' that the formula is correct, hence the last question requires students to prove the formula is true by process of induction. The proof provides significant scaffolding. The subsequent activity on Tetrahedral numbers provides no such scaffolding for the proof. Students should be encouraged to use the proof from this activity to tackle the tetrahedral numbers proof.

The student handout for this activity is provided in two formats:

Version 1: Complete activity with visuals to support the development of the formula, calculator instructions as appropriate and scaffolding with regards to the proof by induction.

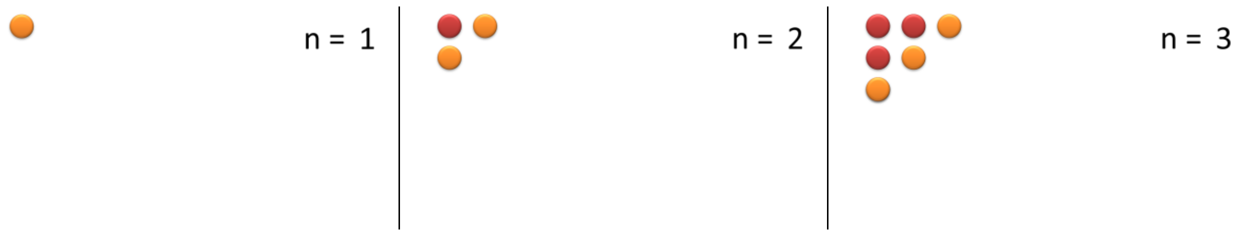
Version 2: An economically print friendly version that is designed to be used with the corresponding PowerPoint slides. The slides contain the visuals to support the development of the formula and calculator instructions. The result is a more teacher lead activity as students rely on the visual presentation. The questions and answers are still the same. The PowerPoint slides also contain the questions in the event that teachers do not wish to print any student handouts.

What is the sum of the first *n* whole numbers?

There are several ways this problem can be solved. Given any value for *n* you could add the numbers up, one by one, but if *n* is large this could take a lot of time. A formula would be a much quicker way to determine such a sum. In this activity you will work with a range of visual and numerical methods to arrive at a formula. However, the formula is based on observation, intuition and a relatively small sample of numbers. There are many cases where formulas seemed to work, but are later found to be flawed. In the final stage of this activity you will prove that your formula works for all whole number values for *n*.

Visual Observation

The series of diagrams below show one way to visualise the sum of the first *n* whole numbers. In each case the new row (orange) shows the quantity being added. The diagrams show why the pattern is referred to as 'triangular' numbers. The last representation includes a duplication of the pattern.



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Question: 1.

The following questions refer to the last pattern (n = 6).

a) How many dots in the last pattern?
Answer: 42

b) Explain how you determined this quantity.
Answer: (Answers will vary) – Multiplied the number of rows by the number of columns: 6 x 7 = 42

c) What is the sum of the first 6 whole numbers?
Answer: 21

Question: 2.

Determine the sum of the first 7 whole numbers without using 'addition'.
Answer: A rectangle of 7 x 8 dots could be drawn. Half of this value would be the sum of the first 7 whole numbers. Therefore 7 x 8 ÷ 2 = 28. [Sum of the first 7 whole numbers]

Question: 3.

Generalise your answer to Question 2 for the sum of the first *n* whole numbers.
Answer: Sum of first n whole numbers = n x (n + 1) ÷ 2

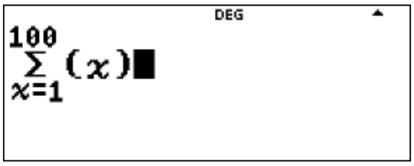
Question: 4.

Use your formula (above) to calculate the sum of the first 100 whole numbers.
Answer: Sum = 100 x 101 ÷ 2 = 5050

Question: 5.

Use the sum command on your calculator to determine the sum of the first 100 numbers.

Expression: $\sum_{x=1}^{100} x$ Calculator instructions: Press **[math]** and select option: **5** sum(.



Answer: 5050

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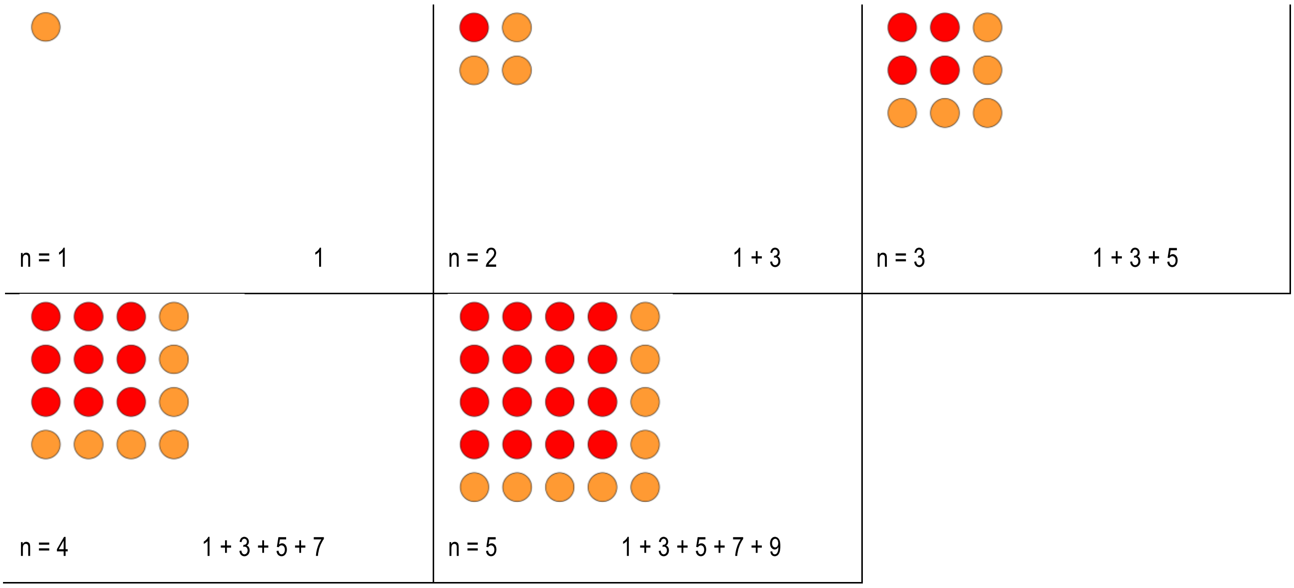
Visual Observation + Numerical Intuition:

In this section you will study the sum of the first *n* odd numbers, then the first *n* even numbers and finally derive a formula for the sum of the first *n* whole numbers.

The following sequence of images represents the sum of the first *n* odd numbers.



Question: 6.

The following questions relate to the sum of the first *n* odd numbers.

a) What shape can be formed by the sum of the first *n* odd numbers?
Answer: A square

b) Write a formula for the sum of the first *n* odd umbers.
Answer: Sum of first *n* odd numbers = n^2

c) Use your calculator to check the sum of the first 50 odd numbers by using: $\sum_{x=1}^{50} 2x-1$

Answer: 2500 which is equal to 50^2

Question: 7.

The sum of the first *n* even numbers can be determined by comparing with the sum of the first *n* odd numbers.

1 + 3 + 5 + 7 + 9 + 11 + 13 …

2 + 4 + 6 + 8 + 10 + 12 + 14 …

Notice that each of the even numbers is ‘1’ more than the corresponding odd number.

a) Based on this information, determine the sum of the first 50 even numbers.
Answer: From Q6(c) the sum of the first 50 odd numbers = 2500. There are 50 numbers, each even number is 1 more than the corresponding odd number. So the sum of the first 50 even numbers will be 2500 + 50.

b) Write a formula for the sum of the first *n* even numbers.
Answer: Sum of first *n* even numbers = $(n^2 + n)$

c) Use your calculator to check the sum of the first 50 even numbers by using: $\sum_{x=1}^{50} 2x$

Answer: 2550

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d) If sum of the first n even numbers: 2 + 4 + 6 + …is divided 2, the result is 1 + 2 + 3 … hence write a formula for the sum of the first n whole numbers.

Answer: Sum of first n whole numbers = $\frac{n^2+n}{2}$

Pascal's Triangle – Hidden Gem

Pascal's triangle also contains the triangular numbers.

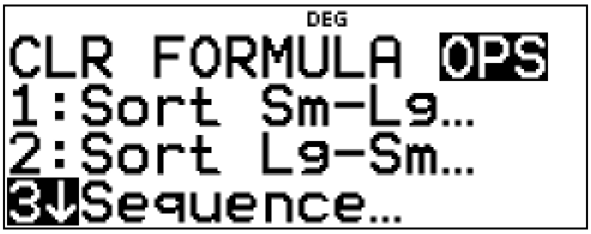
Notice: The n^{th} triangular number is in the $(n+1)^{th}$ row[1].

Example: The number 15 is the 5th triangular number and it is located in the 6th row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Recall that the elements in Pascal's triangle can be computed using combinatorics: ${}^nC_r = \frac{n!}{(n-r)!r!}$

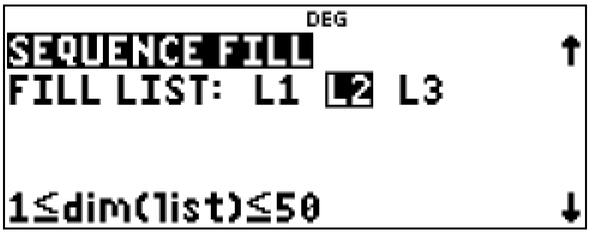
Based on this information, the calculator can be used to generate the triangular numbers.



Select option 3 – Sequence:

The list of triangular numbers will be stored in List 2.

Select List 2:



The expression for List 2 is: ${}^xC_2$

Expression:

Start:

End:

Step Size:

Navigate down to Sequence Fill:

[1] Row numbering in Pascal's triangle starts at row(0) = {1}, row(1) = {1, 1}, row(2) = {1, 2, 1}

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This will calculate: ${}^{2}C_2$, ${}^{3}C_2$, ${}^{4}C_2$, ${}^{5}C_2$ … ${}^{10}C_2$

Question: 8.

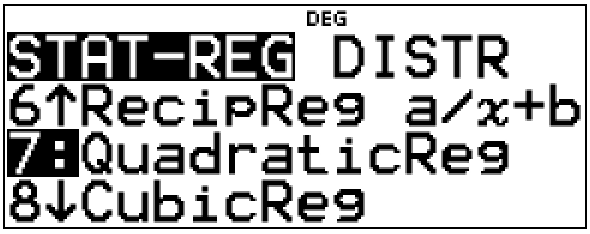
a) Use your calculator to write down the first 9 triangular numbers. [Store the values in L_2]
Answer: Using the process outlined above: {1, 3, 6, 10, 15, 21, 28, 36, 45}

b) Use combinatorics to calculate the 100th triangular number, the sum of the first 100 whole numbers.
Answer: ${}^{101}C_2 = 5050$ [Note that students may inadvertently calculate : ${}^{100}C_2 = 4950$, the 99th triangular number.

c) Store the numbers: {1, 2, 3... 9} in L_1 and use Quadratic regression to determine an equation relating L_1 to L_2. Write down the regression equation. [Calculator Instructions Below]
Answer: $y = 0.5x^2 + 0.5x$

Check to make sure L_1 and L_2 are the same length and corresponding values are aligned.

Access the statistics menu:



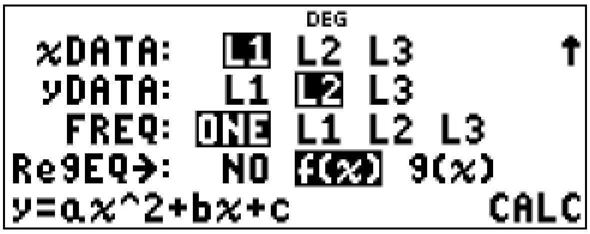
Scroll down to Quadratic Regression: (Option 7)

Match the selections shown opposite then select "CALC".

Note the format for the equation:

$$y = ax^2 + bx + c$$

The regression equation will be stored in $f(x)$.



The equation stored in $f(x)$ can be tested by generating a table of values.

Press the **[Table]** key and select option 1: Add/Edit func. The function will be displayed.

Use the arrow keys to navigate down through the menu to 'calc' then press **[Enter]** to generate the table.

d) The diagonal for the triangular numbers can be written using combinatorics: ${}^{n+1}C_2 = \dfrac{(n+1)!}{((n+1)-2!)2!}$

Simplify this formula to write an expression for the nth triangular number.

Answer: Given the nth triangular number is in the $(n+1)$th row, and the third leading diagonal (r = 2)

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$$\frac{(n+1)!}{(n+1-r)!r!} = \frac{(n+1)!}{(n+1-2)!2!}$$
$$= \frac{(n+1)!}{(n-1)!2}$$
$$= \frac{(n+1)(n)(n-1)(n-2)...}{2(n-1)(n-2)...}$$
$$= \frac{n(n+1)}{2}$$

Induction

The formula for the sum of the first n whole numbers has been generated three different ways. In each case the formula has been based on 'observation', not proof.

Step 1: Show true for n = 1

We must first prove that the formula is true for n = 1.

The sum of the first '1' whole numbers is equal to 1 and $\frac{n(n+1)}{2} = \frac{1\times 2}{2} = 1$

Step 2: Assume true for n

That is: 1 + 2 + 3 + … n = $\frac{n(n+1)}{2}$ **-- Equation 1**

Step 3: Show true for n + 1.

We know that the LHS = (1 + 2 + 3 + … n) + (n + 1),

From Equation 1 we can re-write this as: $\frac{n(n+1)}{2} + (n+1)$

Question: 9.

Complete step 3 by re-writing the RHS: $\frac{(n+1)(n+1+1)}{2}$ to show that: $\frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+1+1)}{2}$

$$RHS = \frac{(n+1)(n+1+1)}{2}$$
$$= \frac{(n+1)(n+2)}{2}$$
$$= \frac{n(n+1)+2(n+1)}{2}$$
$$= \frac{n(n+1)}{2} + (n+1)$$

LHS = RHS as required.

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