Torricelli's Law

ID: 10080

Time required
45 minutes

Activity Overview

Students will examine the velocity of water flowing through the tap of a tank. They will find an equation to model the height of the water in the tank as the tank is drained. As an extension, students will find the trajectory equation for the water as it exits the tank.

Topic: Applications of Integration

- *Given a function that expresses the velocity or acceleration of a moving object as a function of time, integrate to find a function that describes the displacement as a function of time, e.g., Torricelli's Law.*

Teacher Preparation and Notes

- *The students should already be familiar with the concept of the integral, the use of the chain rule, and relationship of displacement, velocity and acceleration. Review the energy equations either before the exercises or during the presentation.*
- *Students should know how to use the* ***integral*** *command.*
- *This activity is designed to be teacher-led.*
- *This program in the extension of this activity requires the use of CAS technology.*
- ***To download the student and solution TI-Nspire documents (.tns files) and student worksheet, go to education.ti.com/exchange and enter "10080" in the keyword search box.***

Associated Materials

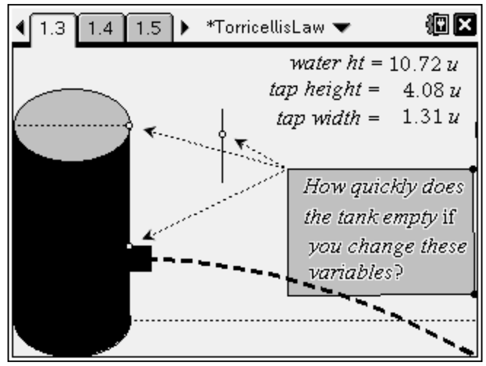
- *TorricellisLaw_Student.doc*
- *TorricellisLaw.tns*
- *TorricellisLaw_Soln.tns*
- *TorricellisLaw_Extension.tns*

Suggested Related Activities

To download any activity listed, go to education.ti.com/exchange and enter the number in the keyword search box.

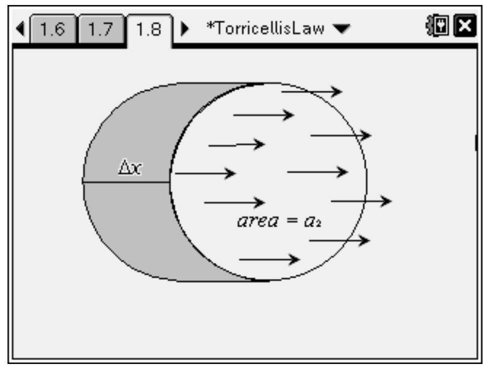
- *Differential Equations (TI-Nspire CAS technology) — 8998*
- *Time Derivatives (TI-Nspire CAS technology) —9537*
- *Hourglass (TI-89 Titanium) — 3458*

Begin by going over Torricelli's Law and the image of the tank. Keep in mind that the equation only holds true if there is no friction or other factors that may slow liquids from exiting the tank. In Physics, there is generally a constant added to the equation to account for this, depending on the liquid and tank.

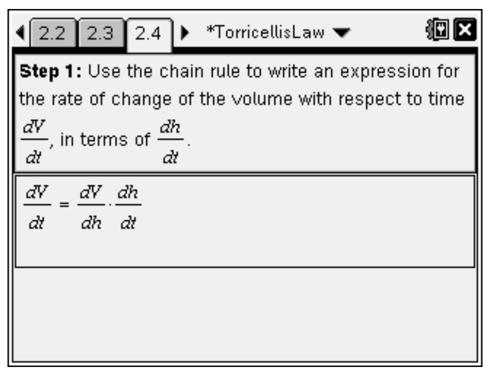


If students are confident with Physics, then you may wish to have them derive Torricelli's Law from the kinetic energy and potential energy equations. These equations can be set equal to each other due to the conservation of energy in a closed system.

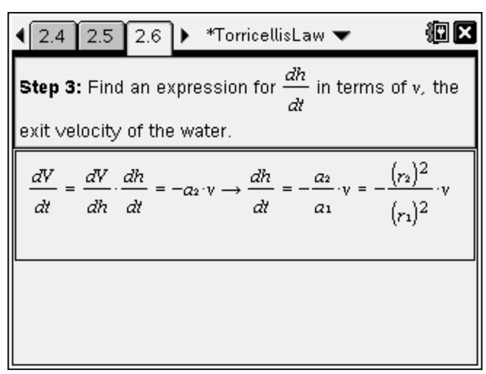
Before starting Step 1, help students make the connection between Δx and the change in volume. As the amount of water that exits the tank in the cross section of the tap is the same amount the volume is decreasing in the tank, so we say that the change in volume is related to the change in height, instead of Δx. This concept is introduced using Δx so that students can make the connection between the water that is displaced (x in kinematic equations) to how quickly it is being displaced (the velocity of the water as it exits the tank).



Link Step 1 to a discussion of the differential equation for the change of volume with respect to time. This expression is linked to the change of the height of the fluid and to both the speed at which the fluid escapes and the diameter of the tap.

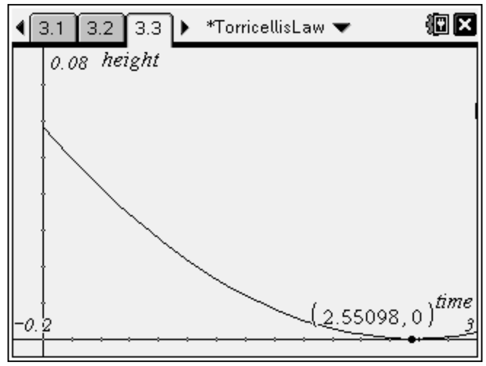


In Step 3, students should use the equation from Step 2 and $\frac{dV}{dt} = -a_2 v$ from the given information. Students can set these equations equal to each other and solve for $\frac{dh}{dt}$.



Students should be given opportunities to discuss and become comfortable with these ideas before proceeding with the application of the chain rule which follows. They should be encouraged to be systematic in identifying all the relevant parameters and functions which apply to such a system and how these relate to each other.

Through application of the chain rule and identifying the relationship between volume of a cylinder and height of the container. Students should be able to build the relationship between time and height for this system, leading to a formula for height with respect to time. They can see the graph of $h(t)$ on page 3.3.



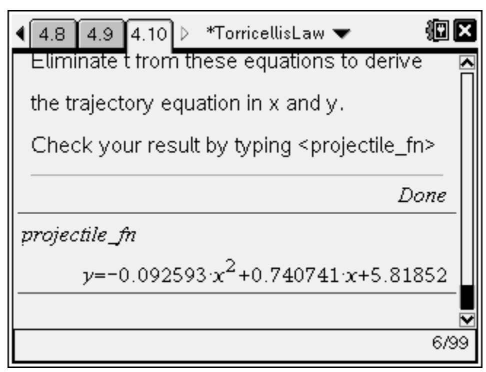
It is critical that students are able to apply what they have learned to the real world situation it describes, and the graphical representation of the height/time formula is an important contributor to this understanding.

Students must be encouraged to clearly describe in their own words how the various components interact in this system, particularly the rates of change that are involved.

Extension

The extension activities involve derivation of the trajectory path for the escaping water using the principles of projectile motion—once again, students are led, step by step, through the process in order to support them in developing such solutions in the future. They are then led through the opportunity to check their solution using a prepared program, **projectile(initial_x, initial_y, initial_velocity, initial_angle, gravity_constant)**. This may be applied to the three different taps given in the model on page 4.6, allowing students the chance to consolidate their application of this important process.

For the three taps, students should find the following equations:



$$y = h_2 - \frac{(x - 2r_1)^2}{4h_1}$$

Tap 1: $h_1 = 8$; $h_2 = 2$; $y = -0.03125x^2 + 0.25x + 1.5$

Tap 2: $h_1 = 5.5$, $h_2 = 4.5$
$y = -0.045455x^2 + .363636x + 3.77273$

Tap 3: $h_1 = 2.7$, $h_2 = 7.3$
$y = -0.092593x^2 + 0.740741x + 5.81852$

Student Solutions

Step 1: $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

Step 2: Since $V = \pi \cdot r^2 \cdot h \rightarrow \frac{dV}{dh} = \pi \cdot r^2 = a_1$, the area of the cross-section of the container.

So we have $\frac{dV}{dt} = a_1 \cdot \frac{dh}{dt}$.

Step 3: $\frac{dV}{dt} = a_1 \cdot \frac{dh}{dt} = -a_2 \cdot v \rightarrow \frac{dh}{dt} = \frac{-a_2}{a_1} \cdot v$

Step 4: $\frac{dh}{dt} = \frac{-a_2}{a_1} \cdot v = \frac{-r_2^2}{r_1^2} \cdot v$

Step 5: $\frac{dh}{dt} = \frac{-r_2^2}{r_1^2} \cdot \sqrt{2 \cdot g \cdot h}$

Step 6: $2\sqrt{h} = \frac{-r_2^2}{r_1^2} \cdot \sqrt{2 \cdot g} \cdot t + C$

Step 7: $h = \left(\frac{-r_2^2}{r_1^2} \cdot \sqrt{2 \cdot g} \cdot t + 2\sqrt{h_1} \right)^2$

Extension

a. When $t = 0$:

Displacement: $x = 2r_1$, $y = h_2$ (where r_1 is the radius of the cylinder and h_2 the height of the tap above the ground)

Velocity: $v_x = \sqrt{2gh_1}$ and $v_y = 0$

Acceleration: $a_x = 0$ and $a_y = -g$.

b. Horizontal components:

Acceleration: $a_x = \frac{d^2}{dt^2}(x) = 0$

Velocity: $v_x = \int a_x \cdot dt = C_1 = \sqrt{2gh_1}$

Displacement: $x = \int v_x \cdot dt = t \cdot \sqrt{2gh_1} + C_2$

But when $t = 0$, $x = C_2 = 2r_1$ and so

$x = 2r_1 + t \cdot \sqrt{2gh_1} \rightarrow t = \frac{x - 2r_1}{\sqrt{2gh_1}}$

c. Vertical components:

Acceleration: $a_y = \frac{d^2}{dt^2}(y) = -g$

Velocity: $v_y = \int a_y \cdot dt = C_3 - g \cdot t = 0$ when $t = 0$, hence $C_3 = 0$.

$v_y = -g \cdot t$

Displacement: $y = \int v_y \cdot dt = C_4 - \frac{1}{2} g \cdot t^2 = h_2$ when $t = 0$ hence $C_4 = h_2$

$y = h_2 - \frac{1}{2} g \cdot t^2$

d. Eliminating t gives $y = h_2 - \frac{1}{2} g \cdot \frac{(x - 2r_1)^2}{2gh_1} = h_2 - \frac{(x - 2r_1)^2}{4h_1}$

e. Tap 1: $y = -0.03125x^2 + 0.25x + 1.5$

Tap 2: $y = -0.045455x^2 + .363636x + 3.77273$

Tap 3: $y = -0.092593x^2 + 0.740741x + 5.81852$