Circular Probability

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Teacher Notes and Answers

7   8   9   10   **11**   12

TI-Nspire™   Investigation   Student   50 min

Problem to Solve

A point P is placed inside a unit square: ABCD. If point P is able to move around randomly inside the square, what is the probability that the angle: $\angle APB \geq 90°$.

The diagram opposite shows a sample point where $\angle APB = 67°$.
The slider can be used to generate points to gain an insight to the problem.
Open the Prob Challenge document to explore this probability.
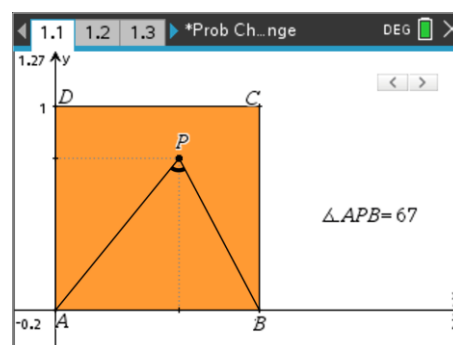
Question: 1.

Click on the slider 10 times and record the quantity of successes ($\angle APB \geq 90$) and failures ($\angle APB < 90$).
Use this data to provide an approximate estimate for the probability.

Answer: Answers will vary around the exact answer of $\dfrac{\pi}{8}$

Building a Simulation

A Spreadsheet application has been inserted in Page 1.2 of the TI-Nspire "Prob-Challenge" document.

Navigate to the formula bar in the first column (data) and generate an automatic data capture of the angle (APB) on Page 1.1.

[[menu] > Data > Data Capture > Automatic

From the variable menu [var], select variable APB or type: APB

Each time variable APB is changed on Page 1.1, the results will be recorded automatically.

The next step is to calculate the estimated probability based on the 'data'.
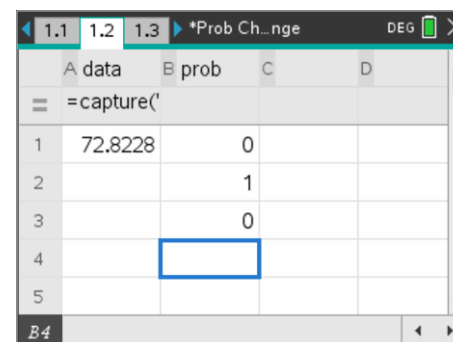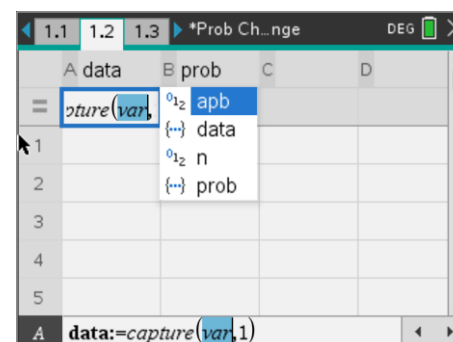
In cell B1 type the formula:

$=countif(data,? \geq 90)$

In cell B2 type the formula:

$=dim(data)$

In cell B3 type the formula:

$=\dfrac{b1}{b2}$

The formula in cell B1 will count how many times the angle in the data column is greater than (or equal) to 90°. The formulate in cell B2 counts how many data points have been collected (dimension of the data list) and cell B3 calculates the proportion of successes from the total number of trials. To see an approximate value for this result the formula can be changed to: =approx(b1/b2).

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Question: 2.

Click on the slider repeatedly (or animate) to collect at least 100 data points. Check the spreadsheet and record the estimate for the probability of success.

Answer: Answers will vary around the exact answer of $\dfrac{\pi}{8}$ but should be a much better approximation than the 10 samples collected in Question 1.

Theory

The theoretical probability is related to Thales theorem. A circle has been drawn with centre at (0.5, 0) passing through (0, 0) and (1, 0). Point P is on the circumference of the circle.
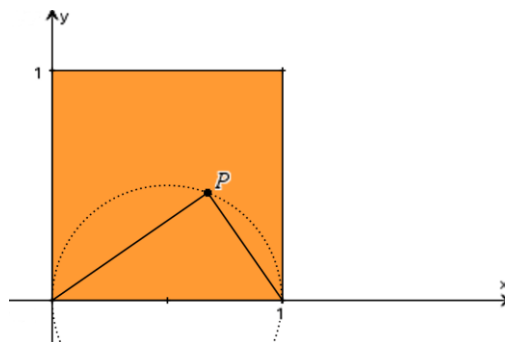
Question: 3.

What is the angle, according to Thales theorem at point P?

Answer: 90°

Question: 4.

Use your answer to Question 3 to determine the proportion of the area enclosed by the semi-circle within the unit square; and hence the probability that for the randomly generated point P is such that: $\angle$ APB $\geq$ 90°.

Answer: The area of the square is 1 unit. The area of the semi-circle of radius ½ is $\dfrac{\pi}{8}$, so the probability is $\dfrac{\pi}{8}$.

Question: 5.

Compare the exact answer (Question 4) and the approximate one obtained using the simulation.

Answer: The probability generated using simulations is a good approximation for the exact answer.

Teacher Notes: The purpose of this last question is to encourage students to use the simulation results as a guide for the extension question. Students should be able to determine a very approximate probability by considering the proportion of the square that is covered by the specified angle. The simulation however provides an even better estimate of the actual answer. Note that the suggested angles to initiate the extension mean that students should be able to determine exact probabilities by using appropriate trigonometric ratios.

A teacher TNS file is also available to help provide students with a 'clue'. Open the Teacher Demonstration file. On Page 1.1 the diagram allows point P to be moved anywhere. The angle can be locked using the attributes menu. (Right click on the angle, select attributes and then the padlock.) Once the angle is locked point P will only be able to move along a path that contains the specified angle. Use the Trace (Geometry) option to track the movement of point P. This will allow students to see that the points lie on a circle. Students may also determine the radius (and centre) of the circle, the teacher can then use this information to see that the circle moves through the traced points. Upon seeing the trace, students may recognise this as

Thales theorem when $\angle$APB = 90° and the central angle theorem for *most* angles other than 90°.
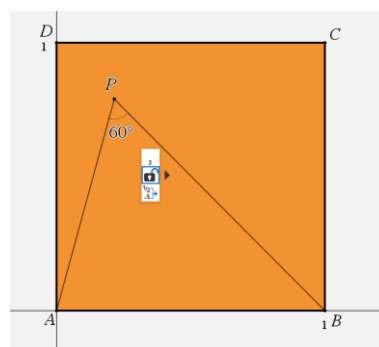
The slider allows various construction clues to be displayed to students. Remember to 'unlock' the angle!

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Extension

In this part of the investigation, ∠ APB can be an angle other than 90°. Start by exploring specific angles (see below) and work towards a general formula for the probability for an angle x such that $\angle APB \geq x$. In each case a probability estimate can be obtained by changing the corresponding spreadsheet formula. The data can be cleared out if desired.

To clear the existing data, navigate to the top of the data column (Column A) and keep on arrowing upwards until the column is highlighted.

[menu] > Data > Clear Data

This will clear data without deleting the automatic data capture command in the formula bar.

Change the formula in cell B1 then return to Page 1.1 to generate data.
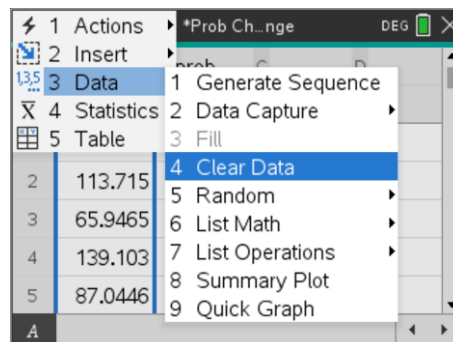
Explore probabilities for each of the following:

Angle:	Simulation	Theoretical:
45°	1	1
60°	≈ 0.78 (200 simulations)	$\dfrac{\pi}{9} - \dfrac{\sqrt{3}}{4} \approx 0.7821$
120°	≈ 0.22 (200 simulations)	$\dfrac{\pi}{9} - \dfrac{\sqrt{3}}{12} \approx 0.2047$

Determine the general probability for an angle $x°$. [Hint: Consider all scenarios]

Answer: 60°

∠APB lies on an arc that passes through points A and B and centre Q.

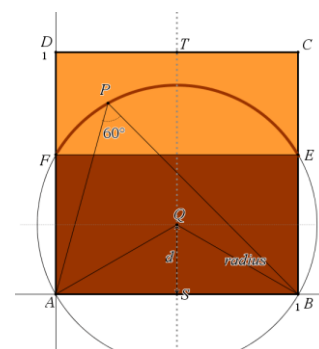∠AQB is the angle at the centre with ∠APB, which means ∠AQB = 2 x ∠APB = 120°.

Let S be the midpoint of AB forming a right angled triangle: △ ASQ.

$AS = \dfrac{1}{2}$ therefore $QS = \dfrac{1}{2\sqrt{3}}$ and $AQ = \dfrac{1}{\sqrt{3}}$

The proportion of the square where ∠APB ≥ 60° is equal to the area of the segment formed by the arc EF (centred at Q) plus the area of the rectangle ABEF.

$$Area = \dfrac{1}{2}\left(\theta^r - \sin(\theta)\right)r^2 + length \times width$$

$$= \dfrac{1}{2}\left(\dfrac{120\pi}{180} - \sin(120)\right)\left(\dfrac{1}{\sqrt{3}}\right)^2 + 1 \times 2 \times \dfrac{1}{2\sqrt{3}} = \dfrac{\pi}{9} + \dfrac{\sqrt{3}}{4}$$

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Answer: $\angle APB = 120°$
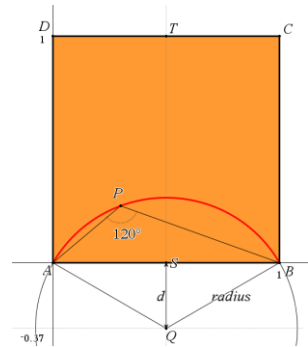
$\angle APB$ lies on an arc that passes through points A and B and centre Q.

$\angle AQB = 120°$. $[360° - 2\angle APB = 120°]$

$AS = \dfrac{1}{2}$ therefore $QS = \dfrac{1}{2\sqrt{3}}$ and $AQ = \dfrac{1}{\sqrt{3}}$

Proportion of the square where $\angle APB \geq 120°$ is equal to the area of the segment formed by the arc AB (centred at Q)

$$Area = \frac{1}{2}\left(\theta^r - \sin(\theta)\right)r^2$$

$$= \frac{1}{2}\left(\frac{120\pi}{180} - \sin(120)\right)\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{\pi}{9} - \frac{\sqrt{3}}{12} \approx 0.2047$$

Answer: 45°

This is the simplest answer of all. When $\angle APB = 45°$, point P passes through C and D so all angles within the unit square are greater than 45°.

General Answer: [Assuming the angle is measured in degrees]

Angles greater than 90°

Let $\theta = \angle APB$ where $\theta \geq 90°$ the entire favourable area is a segment.

The angle for the sector is: $360 - 2 \times \angle APB$

$AS = \dfrac{1}{2}$ and $AQ = \dfrac{1}{2\sin\left(\dfrac{360-2\theta}{2}\right)} = \dfrac{1}{2\sin(\theta)}$ (radius) and $QS = \dfrac{1}{2\left|\tan\left(\dfrac{360-2\theta}{2}\right)\right|} = \dfrac{1}{2\tan(\theta)}$

Note that the absolute value of the tangent function is included as QS is a length.

The area of the segment and therefore the probability $= \dfrac{1}{8\sin^2(\theta)}\left(\dfrac{180-\theta}{90}\pi + \sin(2\theta)\right)$

This general formula can be checked using $\theta = 90°$, $\theta = 120°$ and other simulated values.

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Angles between 2Tan⁻¹(½) and 90°

When **2Tan⁻¹(½) ≤ ∠APB ≤ 90°** the required area is the combination of a rectangle and segment. Note that the lower bound limit represents the point when the arc APB goes beyond the bounds of the unit square producing a new combination that students may not have realised.

Probability for any angle specified within **2Tan⁻¹(½) ≤ ∠APB ≤ 90°:**

Using the same approach as before:

AS = $\dfrac{1}{2}$ and $AQ = \dfrac{1}{2\sin(\theta)}$ (radius) and $QS = \dfrac{1}{2\tan(\theta)}$

The rectangle has a width of 1 unit and a height of 2 x QS.

The area required is therefore:

$$\frac{1}{8\sin^2(\theta)}\left(\frac{2\theta\pi}{180}-\sin(2\theta)\right)+\frac{1}{\tan(\theta)}=\frac{270\sin(2\theta)+\pi\theta}{360(1-\cos(2\theta))} \quad \text{(CAS simplification)}$$

This general formula can be checked using θ = 90°, θ = 120° and other simulated values. Students should however notice that the area (probability) does NOT equal 1 when θ = 45°. This is where students should return to their diagram and identify the range of angles for which the arc extends beyond the unit square.

Consider: Δ AST or Δ ASP. The triangle has a base of ½ and a height of 1. Once ∠ APB exceeds 2 x Tan⁻¹(½) a range of locations for P will go beyond the unit square. The sector area will therefore include area that is not permitted.

The smallest angle remains as 45° but for angles larger than 45° and smaller than 2 x Tan⁻¹(½) a new rule must be established.

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45° ≤ ∠APB ≤ 2Tan⁻¹(½)
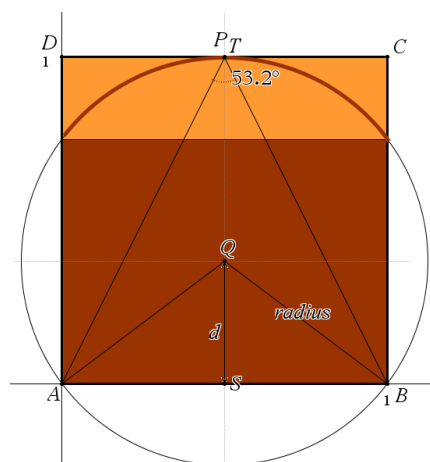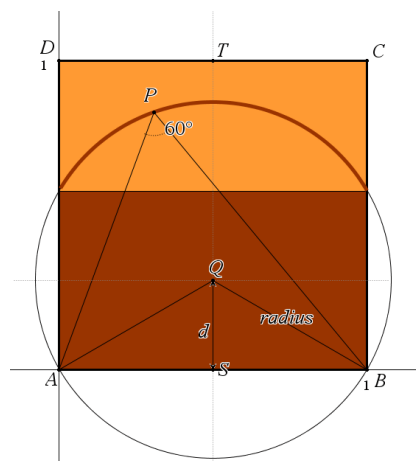
This problem is somewhat more complicated than the previous scenario. The area can be considered as follows:

Area of Rectangle + Area Segment 1 – Area Segment 2.

Segment 1 is the same as that previously computed.

Segment 2 is that protruding beyond the unit square.

The angle required to compute the area segment 2 can be determined by the distance TQ (1 – d) and the radius of the circle.

To help reduce cognitive overload and the immense algebraic and trigonometric simplifications it is useful for students to define functions and areas on the calculator leaving students free to THINK!
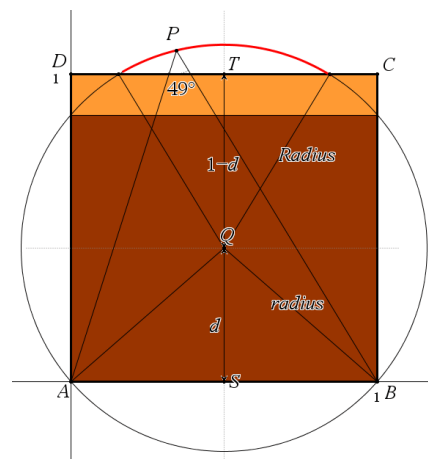
Key expressions:

Radius of (circle) arc: $r = \dfrac{1}{2\sin(\theta)}$ Distance QS: $d = \dfrac{1}{2\tan(\theta)}$ Distance QT: $1 - d$

Angle TQC is required for the smaller segment.

∠ TQC can be computed from: $\cos(TQC) = \dfrac{1-d}{r}$

Now that the angle for the smaller sector is known the difference between the two sectors can be computed and added to the rectangular area. The overall expression is too large to represent here but has been defined on the Teacher demonstration file as a(x) where x represents ∠APB as defined above.

© Define radius (arc), distance d (QS) and also angle TQC for smaller segment

Define $r(x) = \dfrac{1}{2 \cdot \sin(x)}$ *Done*

Define $d(x) = \dfrac{1}{2 \cdot \tan(x)}$ *Done*

Define $tqc(x) = \cos^{-1}\left(\dfrac{1-d(x)}{r(x)}\right)$ *Done*

© Define Areas Rectangle, Sector 1 and Sector 2

Define $rectangle(x) = 2 \cdot d(x)$ *Done*

Define $sector1(x) = \dfrac{1}{8 \cdot (\sin(x))^2} \cdot \left(\dfrac{\pi \cdot x}{90} - \sin(2 \cdot x)\right)$ *Done*

Define $sector2(x) = \dfrac{1}{8 \cdot (\sin(x))^2} \cdot \left(\dfrac{tqc(x) \cdot \pi}{90} - \sin(2 \cdot tqc(x))\right)$ *Done*

Define $area(x) = rectangle(x) + sector1(x) - sector2(x)$ *Done*

$area(45)$ 1.

$area(50)$ 0.972008

$area\left(2 \cdot \tan^{-1}\left(\dfrac{1}{2}\right)\right)$ 0.924725

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Teacher Notes

The teacher solution file demonstrates how the CAS can handle all the background manipulation while students build the diagram and definitions for key components of the problem.

Students can check the result with the angle 45° as this should produce a probability of 1.

Another suitable angle is 2 x Tan^{-1}(½) as students have a formula for this scenario, however students will need to do an 'approximate' answer to avoid potential domain errors.

Students can also check angles within this range but need to understand that probabilities close to 1 will require significantly more simulations to obtain a reasonably accurate answer.

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