Arc length and surface area

Teachers Teaching with Technology™ Professional Development from Texas Instruments

Teacher guide with answers

7 8 9 10 11 **12**

TI-Nspire™ | Activity | Teacher | 6 hours

Introduction

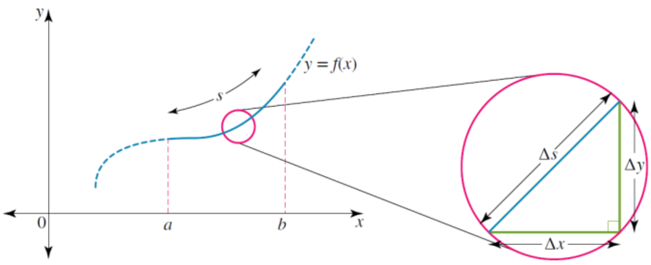
We have used integration and developed results to find areas and areas between curves and volumes of revolution when curves are rotated about either the *x* or *y*-axes. In this investigation we will develop definite integrals to find the length of a plane curve and the surface area formed when the curve is rotated about the *x* or *y*-axes.

Length of a curve

Suppose that the curve $y = f(x)$ is continuous on the closed interval $a \le x \le b$. The curve can be thought of as being made up of infinitely many short line segments as shown.

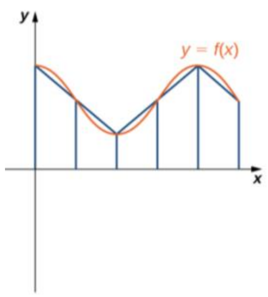
Consider two typical neighbouring points with coordinates $P(x_{i-1}, f(x_{i-1}))$ and $Q(x_i, f(x_i))$ on the curve. Let $\Delta x = x_i - x_{i-1}$ and $\Delta y = f(x_i) - f(x_{i-1})$, then length of the small slope joining the points *P* and *Q* is

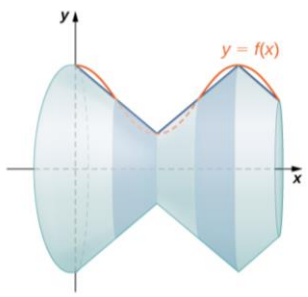
$$\Delta s = d(PQ) = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

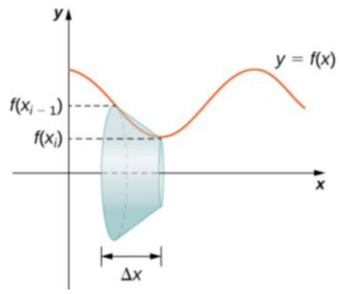


Surface area

When the curve is rotated 360° about the *x*-axis, it forms a surface of revolution S .







The surface area of the disc or frustrum of a cone is given by $\Delta S = \pi\left(f(x_i) + f(x_{i-1})\right)\Delta s$ (this result will be verified later, see example 7), where *s* is the arc length. We can approximate the arc length and surface area from $x = a$ to $x = b$ by subdividing the interval $x \in [a, b]$ into *n* equal strips each of width *h* and then $x_0 = a,\ x_n = b,\ x_i = a + ih,$ where $h = \dfrac{b-a}{n}$.

Author: Raymond Rozen

Question 1

Using the provided tns file, for the function $f(x)=\sqrt{x}$. Subdivide the interval $x\in[0,6]$ into *n* strips, where $n=1,2,3,4,6,10,50,100$ and in each case give your answers correct to four decimal places, to

a. find the length of the curve from $x=0$ to $x=6$.

b. When the curve is rotated about the *x*-axis it forms a volume, find the surface area of this volume of revolution.

c. Are these values approaching a limiting value?

Answer

a.b.

n	arclength	surface area
1	6.4807	49.8712
2	6.5487	59.3709
3	6.5834	61.8905
4	6.6051	62.9688
6	6.6312	63.8874
10	6.6502	64.4706
50	6.6907	64.8976
100	6.6944	64.9180

c. The values appear to be approaching a limiting value.

Question 2

Repeat for the function $f(x)=\log_e(x)$. Subdivide the interval $x\in[1,7]$ into *n* strips, where $n=1,2,3,4,6,10,50,100$ and in each case give your answers correct to four decimal places, to

a. find the length of the curve from $x=1$ to $x=7$.

b. When the curve is rotated about the *x*-axis it forms a volume, find the surface area of this volume of revolution.

c. Are these values approaching a limiting value?

Answer

a.b.

n	arclength	surface area
1	6.3077	72.2446
2	6.3568	75.6876
3	6.3747	76.4226
4	6.3823	76.6920
6	6.3893	76.8886
10	6.6933	76.9903
50	6.3957	77.0453
100	6.3959	77.0470

c. The values appear to be approaching a limiting value.

Author: Raymond Rozen

General results for arc length and surface area

Denoting $\Delta x = x_i - x_{i-1}$, $\Delta y = f(x_i) - f(x_{i-1})$, $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, and

$x_0 = a,\ x_n = b,\ x_i = a + ih,\ h = \dfrac{b-a}{n}$, the total length of the curve s from $x = a$ to $x = b$ is obtained by summing over all such elements and taking the limit as $n \to \infty$ or as $\Delta x \to 0$.

$$s = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\ dx$$

The total surface area S when the curve is rotated about *x*-axis is found by summing over all such discs between $x = a$ and $x = b$ and taking the limit as $n \to \infty$ or as $\Delta x \to 0$.

$$S = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi\left(f(x_i) + f(x_{i-1})\right)\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$S = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \Delta s = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{(\Delta x)^2 + (\Delta y)^2} = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

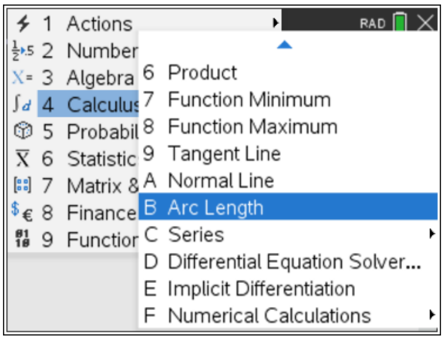
$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = 2\pi \int_a^b y\, ds = 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2}\ dx$$

Note that for some functions the definite integrals obtained for the arc length or surface area can not be found using our integration techniques and CAS must be used to evaluate these definite integrals.

Using CAS for arc length

CAS has a built in function to determine the arc length of the curve $y = f(x)$ over the interval $a \le x \le b$.

On a calculator page, choose 4:Calculus B: Arc Length and complete using the syntax $\text{arcLen}\left(f(x), x, a, b\right)$.



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Question 3

Setup a definite integral to find the length of the curve $y=x^3+\frac{1}{12x}$ from $x=1$ to $x=2$ and evaluate, giving your answer in the form $\frac{a}{b}$ where $a,b\in Z$. Check your answer using CAS.

Solution

$$y=x^3+\frac{1}{12x}=x^3+\frac{1}{12}x^{-1}$$

$$\frac{dy}{dx}=3x^2-\frac{1}{12}x^{-2}=3x^2-\frac{1}{12x^2}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\left(3x^2-\frac{1}{12x^2}\right)^2$$

$$=1+\left(3x^2\right)^2-2\times3x^2\times\frac{1}{12x^2}+\left(\frac{1}{12x^2}\right)^2$$

$$=1+9x^4-\frac{1}{2}+\frac{1}{144x^4}=9x^4+\frac{1}{2}+\frac{1}{144x^4}$$

$$=\left(3x^2+\frac{1}{12x^2}\right)^2$$

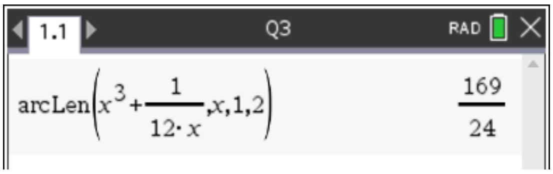
$$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \text{ with } a=1 \text{ and } b=2$$

$$s=\int_1^2\left(3x^2+\frac{1}{12x^2}\right)dx=\int_1^2\left(3x^2+\frac{1}{12}x^{-2}\right)dx$$

$$s=\left[x^3-\frac{1}{12}x^{-1}\right]_1^2=\left[x^3-\frac{1}{12x}\right]_1^2$$

$$s=\left(8-\frac{1}{24}\right)-\left(1-\frac{1}{12}\right)$$

$$s=\frac{169}{24}$$



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Question 4

For curves of the form $y = x^n + \frac{1}{c\,x^m}$ for which we can find the arc length, by evaluating the definite integral by hand, express *m* and *c* in terms of *n*. Given another four types of curves of this form for which we can find the arc length.

Solution

$$y = x^n + \frac{1}{c\,x^m} = x^n + \frac{1}{c}x^{-m} \quad , \quad \frac{dy}{dx} = nx^{n-1} - \frac{m}{c}x^{-m-1} = nx^{n-1} - \frac{m}{cx^{m+1}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(nx^{n-1} - \frac{m}{cx^{m+1}}\right)^2$$

$$= 1+\left(nx^{n-1}\right)^2 - 2\times nx^{n-1}\times\frac{m}{cx^{m+1}} + \left(\frac{m}{cx^{m+1}}\right)^2$$

$$= 1+n^2x^{2n-2} - \frac{1}{2} + \frac{m}{cx^{2m+1}} = n^2x^{2n-2} + \frac{1}{2} + \frac{m}{cx^{2m+1}} = \left(nx^{n-1} + \frac{m}{cx^{m+1}}\right)^2$$

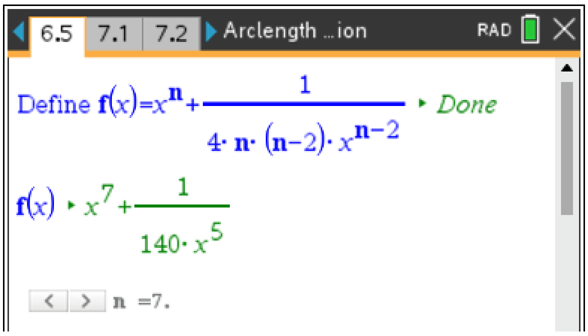
For the middle term to be constant and overall then a perfect square, we require

$$\frac{2nmx^{n-1}}{cx^{m+1}} = \frac{2nmx^{n-m-2}}{c} = \frac{1}{2}$$

$$m = n-2 \quad c = 4nm = 4n(n-2)$$

Functions of the form $y = x^n + \frac{1}{4n(n-2)x^{n-2}}$, $n = 3, 4, 5, 6, ..$

n	$y = f(x)$
3	$y = x^3 + \frac{1}{12x}$
4	$y = x^4 + \frac{1}{32x^2}$
5	$y = x^5 + \frac{1}{60x^3}$
6	$y = x^6 + \frac{1}{96x^6}$
7	$y = x^7 + \frac{1}{140x^5}$



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Question 5

For the function $f\left(x\right)=\sqrt{x}$.

a. Set up a definite integral for the length of the curve from $x=0$ to $x=6$ and using CAS evaluate.

b. When the curve is rotated about the *x*-axis it forms a volume, find a definite integral for the surface area of this volume of revolution giving your answer in the form $\frac{a\pi}{b}$ where $a,b\in Z$.

Solution

a.

$$y=\sqrt{x}=x^{\frac{1}{2}} \quad , \quad \frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$$

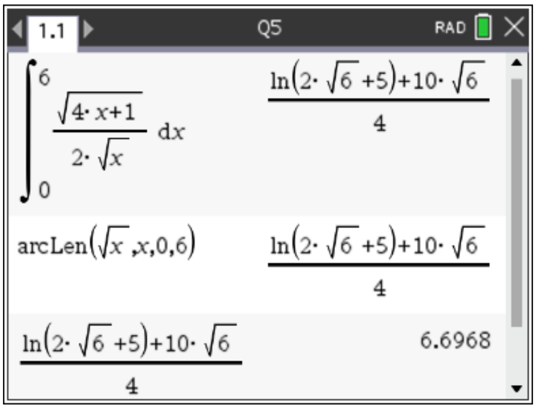
$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{1}{4x}=\frac{4x+1}{4x}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\sqrt{4x+1}}{2\sqrt{x}}$$

$$s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\quad a=0,\quad b=6$$

$$s=\int_0^6\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx$$

$$s=\frac{1}{4}\left[\log_e\left(2\sqrt{6}+5\right)+10\sqrt{6}\right]\approx 6.6968$$



b.

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\quad a=0,\quad b=6$$

$$S=\int_0^6 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\quad a=0,\quad b=6$$

$$S=2\pi\int_0^6\sqrt{x}\,\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx$$

$$S=\pi\int_0^6\sqrt{4x+1}\,dx$$

$$\text{let}\quad u=4x+1\quad \frac{du}{dx}=4$$

$$\text{terminals}\quad x=6,\ u=25,\quad x=0,\ u=1$$

$$S=\frac{\pi}{4}\int_1^{25}u^{\frac{1}{2}}\,du=\frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{25}$$

$$S=\frac{\pi}{6}\left(25^{\frac{3}{2}}-1^{\frac{3}{2}}\right)=\frac{\pi}{6}(125-1)$$

$$S=\frac{62\pi}{3}$$

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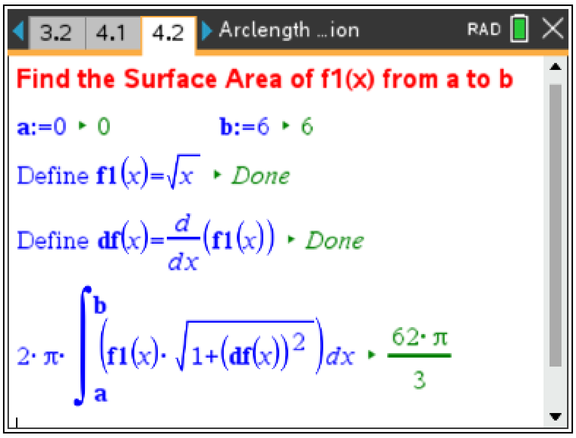
Using CAS for surface area

CAS does not have a built in or intrinsic function to find the surface area, but we can easily develop a notes page to implement this or write a TI-Nspire program.



Question 6

Setup a definite integral to find the surface area obtained by rotating the curve $y=\sqrt{3x+4}$ from $x=0$ to $x=2$ about the *x*-axis and evaluate giving your answer in the form $\frac{a\pi}{b}$ where $a,b\in Z$. Check your result using CAS.

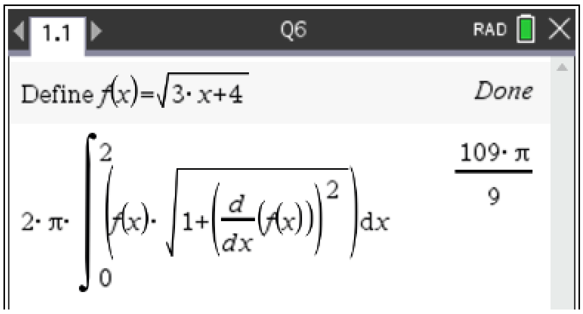
Solution

$$y=\sqrt{3x+4}=(3x+4)^{\frac{1}{2}}\,, \qquad \frac{dy}{dx}=3\times\frac{1}{2}\times(3x+4)^{-\frac{1}{2}}=\frac{3}{2\sqrt{3x+4}}$$

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{3x+4}\sqrt{1+\frac{9}{4(3x+4)}}$$

$$=\sqrt{3x+4}\sqrt{\frac{4(3x+4)+9}{4\sqrt{3x+4}}}=\frac{1}{2}\sqrt{12x+25}$$

$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \text{ with } a=0 \text{ and } b=2$$



$$S=2\pi\int_0^2 \frac{\sqrt{12x+25}}{2}\,dx=\pi\int_0^2 (12x+25)^{\frac{1}{2}}\,dx$$

$$S=\pi\left[\frac{1}{12}\times\frac{2}{3}(12x+25)^{\frac{3}{2}}\right]_0^2$$

$$S=\frac{\pi}{18}\left(49^{\frac{3}{2}}-25^{\frac{3}{2}}\right)=\frac{\pi}{18}(343-125)=\frac{\pi}{18}(218)$$

$$S=\frac{109\pi}{9}$$

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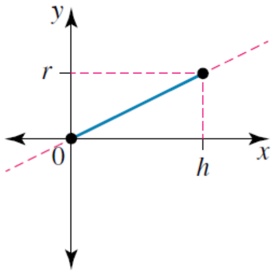
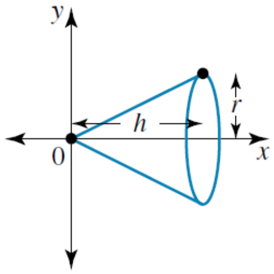
Surface area of a cone

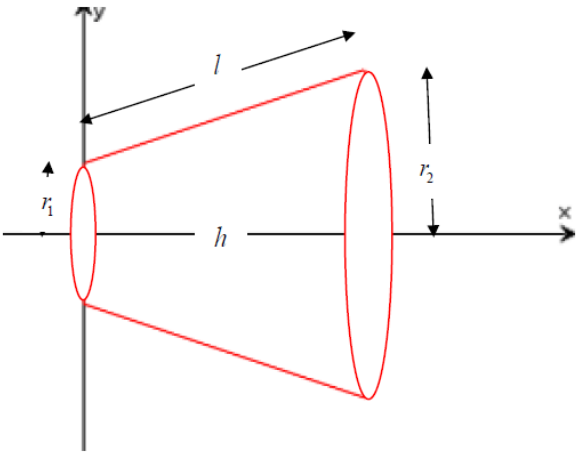




Question 7

a. Show that the surface area of a cone with height *h* and radius *r* is given by $S=\pi r\sqrt{h^2+r^2}$ or $S=\pi rs$ where $s=\sqrt{h^2+r^2}$.

Truncated cone



b. Show that the surface area of a truncated cone (or frustrum) with inner and outer radii of r_1 and r_2 and slope length *l* is given by $S=\pi(r_1+r_2)l$.

Solution

a. the line being rotated about the *x*-axis is $y=mx$

now $m=\frac{r}{h},\quad y=\frac{rx}{h},\quad \frac{dy}{dx}=\frac{r}{h}=m$

$$S=2\pi\int_0^h \frac{rx}{h}\sqrt{1+\frac{r^2}{h^2}}\,dx$$

$$S=\frac{2\pi r}{h}\sqrt{\frac{h^2+r^2}{h^2}}\left[\frac{1}{2}x^2\right]_0^h$$

$$S=\frac{2\pi r\sqrt{h^2+r^2}}{h^2}\left[\frac{1}{2}\left(h^2-0\right)\right]$$

$$S=\pi r\sqrt{h^2+r^2}=\pi rs,\quad s=\sqrt{h^2+r^2}$$

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b. the line being rotated about the *x*-axis is $y=\left(\frac{r_2-r_1}{h}\right)x+r_1$ and $\frac{dy}{dx}=\frac{r_2-r_1}{h}$ also $l^2=\left(r_2-r_1\right)^2+h^2$

$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S=2\pi\int_0^h\left(\left(\frac{r_2-r_1}{h}\right)x+r_1\right)\sqrt{1+\left(\frac{r_2-r_1}{h}\right)^2}\,dx$$

$$S=2\pi\sqrt{\frac{h^2+\left(r_2-r_1\right)^2}{h^2}}\left[\left(\frac{r_2-r_1}{2h}\right)x^2+r_1x\right]_0^h$$

$$S=\frac{2\pi l}{h}\left[\left(\frac{r_2-r_1}{2h}\right)h^2+r_1h\right]$$

$$S=\frac{2\pi l}{h}\left[\frac{1}{2}\left(r_2-r_1\right)h+r_1h\right]$$

$$S=\pi\left(r_1+r_2\right)l$$

This now verifies the result used on page 1.

Arc length with the *y*-axis

Suppose that the curve $x=g\left(y\right)$ is continuous on the closed interval $c\le y\le d$. The total length of the curve $x=g\left(y\right)$ from $y=c$ to $y=d$ is obtained using the result

$$s=\int_c^d\Delta s=\lim_{\Delta y\to 0}\sum_{y=c}^{y=d}\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}=\lim_{\Delta y\to 0}\sum_{x=a}^{x=b}\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}\,\Delta y$$

$$s=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_c^d\sqrt{1+\left(g'\left(y\right)\right)^2}\,dy$$

Surface area around the *x*-axis

When the curve $x=g\left(y\right)$ is rotated 360° about the *x*-axis, between $y=c$ and $y=d$ it forms a surface of revolution S, let $a=g\left(c\right)$ and $b=g\left(d\right)$ then

$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi\int_{y=c}^{y=d}y\,ds=2\pi\int_{y=c}^{y=d}y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=2\pi\int_{y=c}^{y=d}y\sqrt{1+\left(g'\left(y\right)\right)^2}\,dy$$

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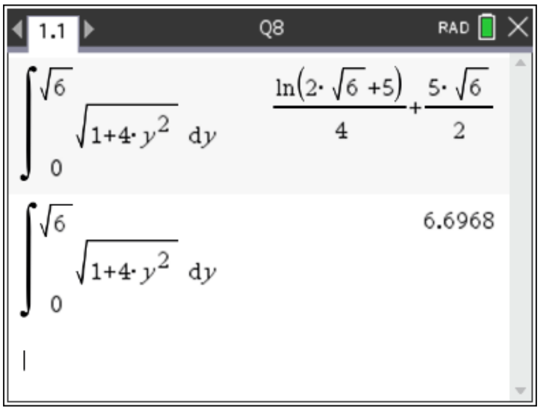
Question 8

For the curve $x = y^2$

a. Setup a definite integral involving *y* to find the length of the curve from $y = 0$ to $y = \sqrt{6}$ and evaluate giving your answer correct to four decimal places, using CAS.

b. When the curve is rotated about the *x*-axis it forms a volume, setup a definite integral involving *y* to find the surface area of this volume of revolution and evaluate, giving your answer in the form $\frac{a\pi}{b}$ where $a, b \in Z$. Check your result using CAS.

c. Compare your results with Question 5.



Solution

a. $x = y^2 \quad \frac{dx}{dy} = 2y$

$$s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dx, \quad c = 0, \quad d = \sqrt{6}$$

$$s = \int_0^{\sqrt{6}} \sqrt{1+4y^2}\, dy$$

$$s = \frac{1}{4}\left[\log_e\left(2\sqrt{6}+5\right)+10\sqrt{6}\right] \approx 6.6968$$

b. $$S = \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy, \quad c = 0, \quad d = \sqrt{6}, \quad S = 2\pi\int_0^{\sqrt{6}} y\sqrt{1+4y^2}\, dy$$

$$\text{let } u = 1+4y^2 \quad \frac{du}{dy} = 8y, \quad \text{terminals} \quad y = \sqrt{6}, \; u = 25, \quad y = 0, \; u = 1$$

$$S = \frac{\pi}{4}\int_1^{25} u^{\frac{1}{2}}\, du = \frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{25} = \frac{\pi}{6}\left(25^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{\pi}{6}(125-1)$$

$$S = \frac{62\pi}{3}$$

c. Answers agree with Question 5.

Arc length and surface area in parametric form

Given the parametric curves $(1)\; x = x(t)$ and $(2)\; y = y(t)$ the length of the curve between $t = t_1$ and $t = t_2$ is given by $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

When the parametric curve is rotated about the *x*-axis, between $t = t_1$ and $t = t_2$ the surface area formed is given by

$$S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Question 9

A particle moves along a curve defined by the vector equation $\underset{\sim}{r}(t)=\cos(t)\underset{\sim}{i}+\cos(2t)\underset{\sim}{j}$ for $0\le t\le\pi$

a. Find the Cartesian equation of the curve and sketch the graph of the curve.

b. Find the speed of the particle.

c. Setup a definite integral involving *t* to find the total length of a curve.

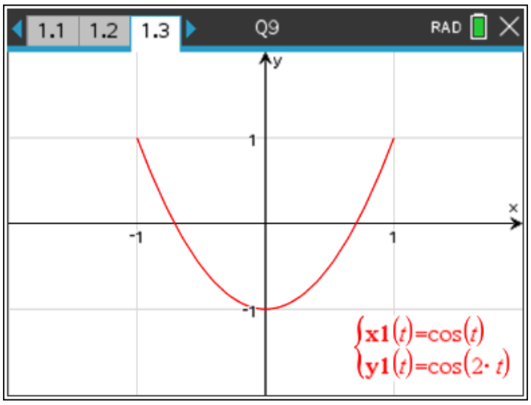
Find the total length correct to four decimal places.

d. When the curve is rotated about the *x*-axis, setup a definite integral involving *t* to find the surface area.

Find the surface area giving your answer in the form $\frac{a\pi}{b}$ where $a,b\in Z$ and check your result using CAS.

e. Verify your results for **c.** using the cartesian equation.

Solution



a. $y=\cos(2t)=2\cos^2(t)-1$, since $t\in[0,\pi]$

$y=2x^2-1$, $x\in[-1,1]$, $y\in[-1,1]$

parabola

b. $x=\cos(t)$ $y=\cos(2t)$

$\dot{x}=\frac{dx}{dt}=-\sin(t)$ $\dot{y}=\frac{dy}{dt}=-2\sin(2t)$

velocity vector $\dot{\underset{\sim}{r}}(t)=\dot{x}(t)\underset{\sim}{i}+\dot{y}(t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t)=-\sin(t)\underset{\sim}{i}-2\sin(2t)\underset{\sim}{j}$

the speed $\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{\sin^2(t)+4\sin^2(2t)}$

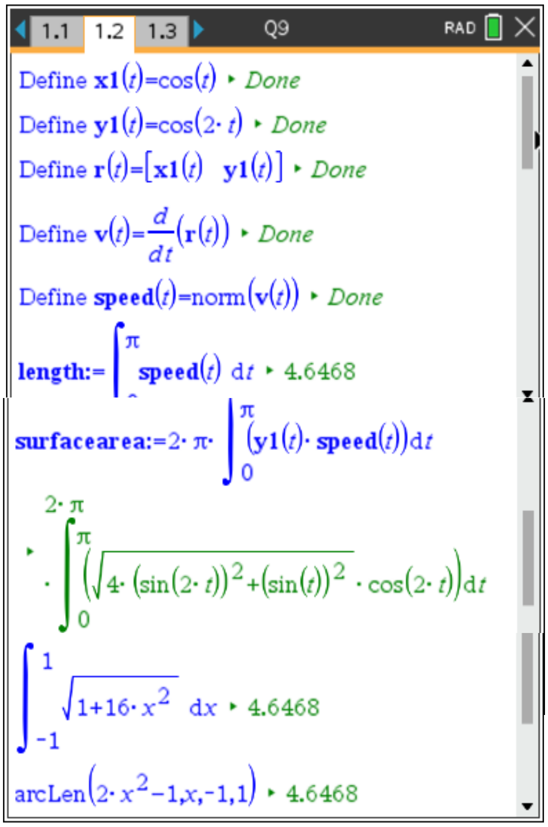
c. $s=\int_0^{\pi}\sqrt{\sin^2(t)+4\sin^2(2t)}\,dt=4.6468$

d. $S=\int_{t_1}^{t_2}2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$S=\int_0^{\pi}\cos(2t)\sqrt{\sin^2(t)+4\sin^2(2t)}\,dt=\frac{64\pi}{3}$

e. $y=2x^2-1$, $\frac{dy}{dx}=4x$

$s=\int_{-1}^{1}\sqrt{1+16x^2}\,dx=4.6468$



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Question 10

A particle moves along a curve defined by the vector equation $\underset{\sim}{r}(t)=(t-\sin(t))\underset{\sim}{i}+(1-\cos(t))\underset{\sim}{j}$ for $0 \le t \le 2\pi$

a. Sketch the graph of the curve using CAS. What is the name of this curve?

b. Find the speed of the particle in terms of *t*.

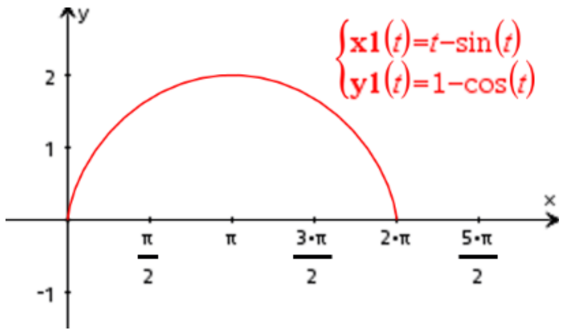
c. Setup a definite integral involving *t* to find the total length of the curve and evaluate.

d. When the curve is rotated about the *x*-axis, setup a definite integral involving *t* to find the surface area and evaluate.

e. Verify your results using CAS to **c.** and **d.**

Solution

a. cycloid



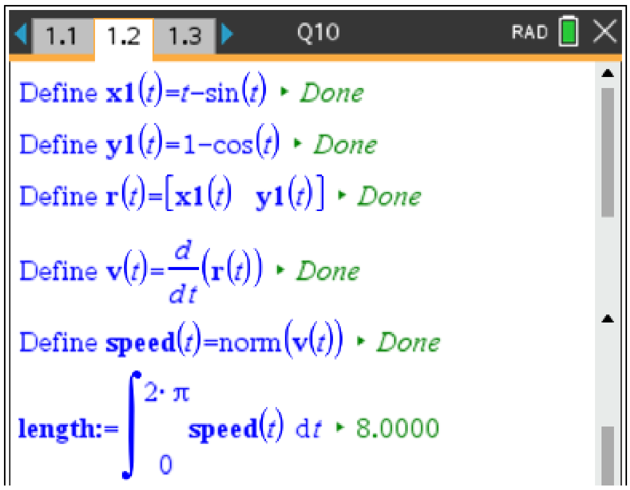
b. $x=t-\sin(t)$, $y=1-\cos(t)=2\sin^2\left(\frac{t}{2}\right)$

$$\dot{x}=\frac{dx}{dt}=1-\cos(t) \qquad \dot{y}=\frac{dy}{dt}=\sin(t)$$

velocity vector $\dot{\underset{\sim}{r}}(t)=(1-\cos(t))\underset{\sim}{i}+\sin(t)\underset{\sim}{j}$

the speed $\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$$\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{(1-\cos(t))^2+\sin^2(t)}$$
$$=\sqrt{1-2\cos(t)+\cos^2(t)+\sin^2(t)}$$
$$=\sqrt{2(1-\cos(t))}=\sqrt{2\left(2\sin^2\left(\frac{t}{2}\right)\right)}$$
$$=2\left|\sin\left(\frac{t}{2}\right)\right|$$
$$=2\sin\left(\frac{t}{2}\right) \quad 0 \le t \le 2\pi$$



c. $$s=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$s=\int_0^{2\pi}2\sin\left(\frac{t}{2}\right)dt$$

$$s=\left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}=-4\cos(\pi)+4\cos(0)$$

$$s=8$$

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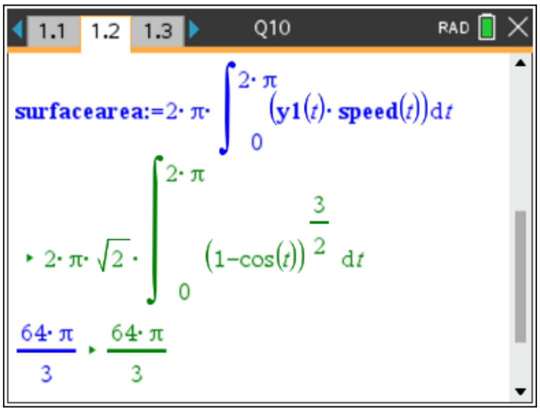
d. $$S=\int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$



$$S=2\pi\int_0^{2\pi} 4\sin^3\left(\frac{t}{2}\right)dt$$

$$S=8\pi\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt$$

$$S=8\pi\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\left(1-\cos^2\left(\frac{t}{2}\right)\right)dt$$

let $u=\cos\left(\frac{t}{2}\right)$ $\frac{du}{dt}=-\frac{1}{2}\sin\left(\frac{t}{2}\right)$

terminals $t=0,\ u=1,\quad t=2\pi,\ u=-1$

$$S=-16\pi\int_1^{-1}\left(1-u^2\right)du=16\pi\left[u-\frac{u^3}{3}\right]_{-1}^{1}=32\pi\left[u-\frac{u^3}{3}\right]_0^1=32\pi\left[\left(1-\frac{1}{3}\right)\right]=\frac{64\pi}{3}$$

Surface area around the *y*-axis

Suppose that the curve $y=f(x)$ is continuous on the closed interval $a\le x\le b$ and let $c=f(a)$ and $d=f(b)$ then When the curve is rotated 360º about the *y*-axis, it forms a surface of revolution S.

$$S=2\pi\int_{x=a}^{x=b} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi\int_{y=c}^{y=d} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Question 11

For the function $f(x)=\sqrt{x},\quad x\in[0,6]$. When the curve is rotated about the *y*-axis it forms a volume, setup two definite integrals to find the surface area of this volume of revolution and evaluate using CAS, giving your answers correct to four decimal places.

Solution

Method 1

$$y=\sqrt{x},\quad \frac{dy}{dx}=\frac{1}{2\sqrt{x}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4x}}=\sqrt{\frac{4x+1}{4x}}$$

$$S=\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,\quad a=0,\quad b=6$$

$$S=2\pi\int_0^6 x\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx$$

$$S=\pi\int_0^6\sqrt{x(4x+1)}\,dx=\frac{1}{32}\left[490\sqrt{6}-\log_e\left(2\sqrt{6}+5\right)\right]$$

$$S\approx 117.6092$$

Method 2

$$y=\sqrt{x},\quad y^2=x\qquad \frac{dx}{dy}=2y$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+4y^2}$$

$$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy,\quad c=0,\quad d=\sqrt{6}$$

$$S=2\pi\int_0^{\sqrt{6}} y^2\sqrt{1+4y^2}\,dy\approx 117.6092$$

Note: Let $x=y^2$ gives Method 1

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Summary of all formulae

Arc Length

x-axis $y = f(x)$ $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$

y-axis $x = g(y)$ $s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_{y=c}^{y=d} \sqrt{1+\left(g'(y)\right)^2}\,dy$

parametric between $t = t_1$ and $t = t_2$, $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

Surface area around the *x*-axis

x-axis $y = f(x)$ $S = \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b y\sqrt{1+\left(f'(x)\right)^2}\,dx$

y-axis $x = g(y)$ $S = 2\pi \int_c^d y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\pi \int_{y=c}^{y=d} y\sqrt{1+\left(g'(y)\right)^2}\,dy$

parametric between $t = t_1$ and $t = t_2$, $S = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

Surface area around the *y*-axis

x-axis $y = f(x)$ $s = \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b x\sqrt{1+\left(f'(x)\right)^2}\,dx$

y-axis $x = g(y)$ $S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = 2\pi \int_{y=c}^{y=d} x\sqrt{1+\left(g'(y)\right)^2}\,dy$

parametric between $t = t_1$ and $t = t_2$, $S = 2\pi \int_{t_1}^{t_2} x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

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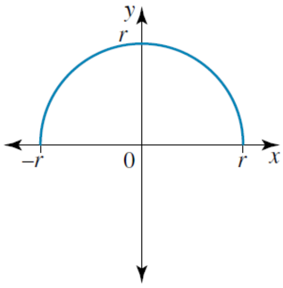
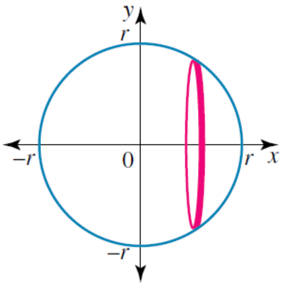
Surface area of a sphere





Question 12

a. Show that the surface area of a sphere of radius *r* is given by $S = 4\pi r^2$.

b. Obtain the parametric equations of a circle of radius *r* and hence verify the result for the surface area of a sphere.

c. Show that the circumference of a circle of radius *r* is $C = 2\pi r$.

Solution

a. $x^2 + y^2 = r^2, \quad y = \sqrt{r^2 - x^2}$

$$2x + 2y\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$S = 2\pi\int_{-r}^{r} y\sqrt{1+\frac{x^2}{y^2}}\, dx = 4\pi\int_{0}^{r} y\sqrt{\frac{x^2+y^2}{y^2}}\, dx$$

$$S = 4\pi\int_0^r r\, dx = 4\pi r\int_0^r 1 dx = 4\pi r\left[x\right]_0^r$$

$$S = 4\pi r^2$$

b. $x = r\cos(t) \qquad y = r\sin(t)$

$\dot{x} = \frac{dx}{dt} = -r\sin(t) \qquad \dot{y} = \frac{dy}{dt} = r\cos(t)$, since $r > 0$

the speed $\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} = \sqrt{r^2\left(\cos^2(t)+\sin^2(t)\right)} = r$

$$S = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$S = 2\pi\int_0^{\pi} r^2\sin(t)\, dt = 2\pi r^2\left[-\cos(t)\right]_0^{\pi} = 2\pi r^2\left(-\cos(\pi)+\cos(0)\right)$$

$$S = 4\pi r^2$$

c. $s = \int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ alternatively $C = \int_{-r}^{r}\sqrt{1+\frac{x^2}{y^2}}\, dx = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\, dx$

using symmetry $C = 4r\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 4r\left(\sin^{-1}(1) - \sin^{-1}(0)\right) = 2\pi r$

$$C = 4\pi\int_0^{\frac{\pi}{2}} r\, dt = 4\pi r\left[t\right]_0^{\frac{\pi}{2}} = 2\pi r$$

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Question 13

For the function $f:[1,7]\to R,\quad f(x)=\log_e(x)$, setup definite integrals involving *x* and use CAS to give answers correct to four decimals, to

a. find the length of the curve from $x=1$ to $x=7$.

b. When the curve is rotated about the *x*-axis it forms a volume, find the surface area of this volume of revolution.

c. When the curve is rotated about the *y*-axis it forms a volume, find the surface area of this volume of revolution.

d. Repeat **a. b.** and **c.** using different definite integrals involving *y*.

Solution

a. $y=\log_e(x),\quad \dfrac{dy}{dx}=\dfrac{1}{x}$

$$x=7,\ y=\ln(7),\quad x=1,\ y=0$$

$$s=\int_1^7\sqrt{\left(1+\left(\frac{1}{x}\right)^2\right)}\,dx$$

$$s=\int_1^7\frac{\sqrt{x^2+1}}{x}\,dx$$

$$s=\log_e\left(\frac{4\sqrt{2}+9}{7}\right)+4\sqrt{2}\approx 6.3959$$

d. $y=\log_e(x),\quad x=e^y,\quad \dfrac{dx}{dy}=e^y$

$$x=7,\ y=\ln(7),\quad x=1,\ y=0$$

$$s=\int_0^{\log_e(7)}\sqrt{\left(1+e^{2y}\right)}\,dy$$

$$s=\log_e\left(\frac{4\sqrt{2}+9}{7}\right)+4\sqrt{2}\approx 6.3959$$

Note let $x=e^y$ gives **a.**

b. $y=\log_e(x),\quad \dfrac{dy}{dx}=\dfrac{1}{x}$

$$S=2\pi\int_1^7 y\sqrt{\left(1+\left(\frac{1}{x}\right)^2\right)}\,dx$$

$$S=2\pi\int_1^7\frac{\log_e(x)\sqrt{x^2+1}}{x}\,dx$$

$$S\approx 49.6326$$

d. $y=\log_e(x),\quad x=e^y,\quad \dfrac{dx}{dy}=e^y$

$$S=2\pi\int_0^{\log_e(7)} y\sqrt{\left(1+e^{2y}\right)}\,dy$$

$$S\approx 49.6326$$

c. $y=\log_e(x),\quad \dfrac{dy}{dx}=\dfrac{1}{x}$

$$S=2\pi\int_1^7 x\sqrt{\left(1+\left(\frac{1}{x}\right)^2\right)}\,dx$$

$$S=2\pi\int_1^7\sqrt{x^2+1}\,dx$$

$$S=\pi\left(\log_e\left(2\sqrt{2}+3\right)+34\sqrt{2}\right)$$

$$S\approx 156.5959$$

d. $y=\log_e(x),\quad x=e^y,\quad \dfrac{dx}{dy}=e^y$

$$S=2\pi\int_0^{\log_e(7)} e^y\sqrt{\left(1+e^{2y}\right)}\,dy$$

$$S=\pi\left(\log_e\left(2\sqrt{2}+3\right)+34\sqrt{2}\right)$$

$$S\approx 156.5959$$

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Question 14

A particle moves along a curve defined by the vector equation $\underset{\sim}{r}(t)=\cos^3(t)\underset{\sim}{i}+\sin^3(t)\underset{\sim}{j}$ for $0\le t\le 2\pi$

a. Sketch the graph of the curve. What is the name of this curve?

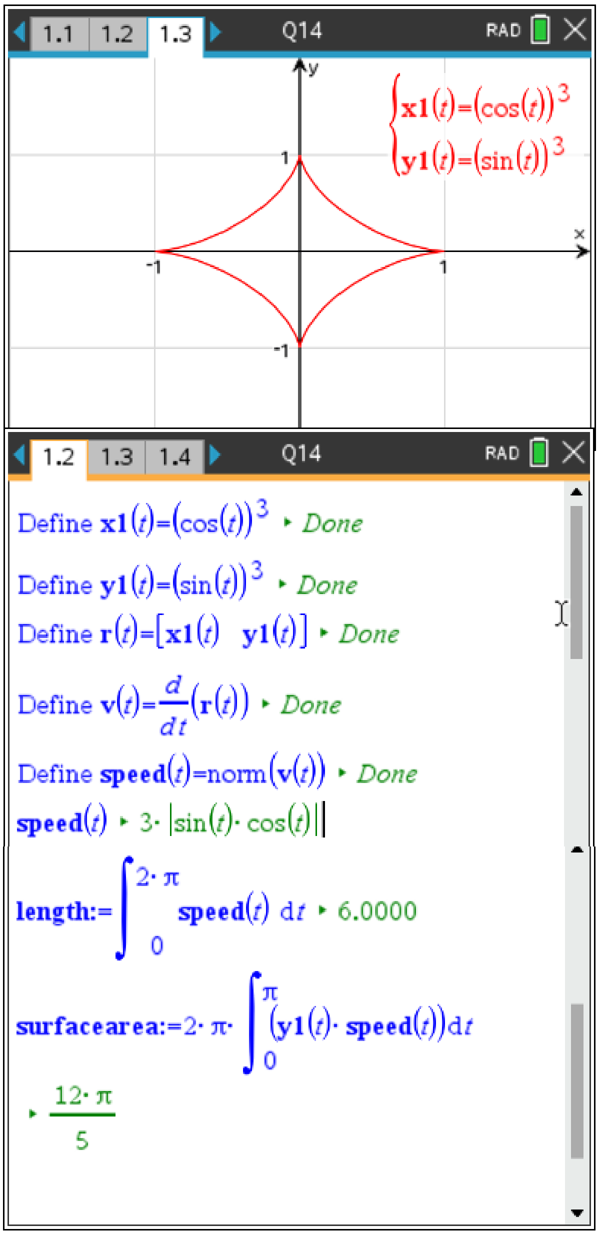
b. Find the speed of the particle.

c. Find the total length of this curve.

d. When the curve is rotated about the *x*-axis, find the surface area.

e. Find an implicit relationship for the equation of the curve and verify your results for **c.** and **d**.

Solution



a. astroid $x=\sin^3(t)$, $y=\cos^3(t)$

b. $\dot{x}=\dfrac{dx}{dt}=-3\cos^2(t)\sin(t)$

$$\dot{y}=\frac{dy}{dt}=3\sin^2(t)\cos(t)$$

the speed $\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}$

$$\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{\left(-3\sin(t)\cos^2(t)\right)^2+\left(3\sin^2(t)\cos(t)\right)}$$

$$=\sqrt{9\sin^2(t)\cos^4(t)+9\sin^4(t)\cos^2(t)}$$

$$=\sqrt{9\sin^2(t)\cos^2(t)\left(\sin^2(t)+\cos^2(t)\right)}$$

$$=\sqrt{9\sin^2(t)\cos^2(t)}=\left|3\sin(t)\cos(t)\right|$$

$$=\frac{3}{2}\left|\sin(2t)\right|=\frac{3}{2}\sin(2t)\quad 0\le t\le\pi$$

c. $s=\displaystyle\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ using symmetry

$$s=4\times\frac{3}{2}\int_0^{\frac{\pi}{2}}\sin(2t)\,dt$$

$$s=6\left[-\frac{1}{2}\cos(2t)\right]_0^{\frac{\pi}{2}}=-3\cos(\pi)+3\cos(0)$$

$$s=6$$

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d. $$S=\int_{t_1}^{t_2}2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$S=2\times 2\pi\int_0^{\frac{\pi}{2}}\sin^3(t)3\sin(t)\cos(t)\,dt$$

$$S=12\pi\int_0^{\frac{\pi}{2}}\cos(t)\sin^4(t)\,dt$$

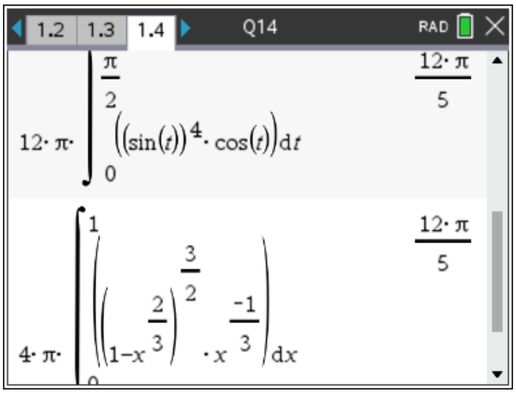
$$\text{let }\ u=\sin(t)\quad \frac{du}{dt}=\cos(t)$$

$$\text{terminals}\quad t=0,\ u=0,\quad t=\frac{\pi}{2},\ u=1$$

$$S=12\pi\int_0^1\left(u^4\right)du$$

$$S=12\pi\left[\frac{u^5}{5}\right]_0^1=12\pi\left[\frac{1}{5}-0\right]$$

$$S=\frac{12\pi}{5}$$



e. $$x=\sin^3(t)\ ,\ y=\cos^3(t)$$

$$\sin(t)=x^{\frac{1}{3}}\ ,\ \cos(t)=y^{\frac{1}{3}},\qquad \sin^2(t)+\cos^2(t)=1$$

$x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$ using implicit differentiation

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0,\qquad \frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}}=\sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}=x^{-\frac{1}{3}}$$ by symmetry

$$s=4\int_0^1 x^{-\frac{1}{3}}\,dx$$

$$s=\frac{4\times 3}{2}\left[x^{\frac{2}{3}}\right]_0^1=6(1-0)$$

$$s=6$$

by symmetry

$$S=2\pi\times 2\int_0^1 y\,x^{-\frac{1}{3}}\,dx$$

$$S=4\pi\int_0^1\left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}x^{-\frac{1}{3}}\,dx$$

$$S=\frac{12\pi}{5}$$

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