Creating Boxes

About the Lesson

In this activity, students will graph the relationship between the length of the sides of cut-out squares and the volume of the resulting box. As a result, students will:

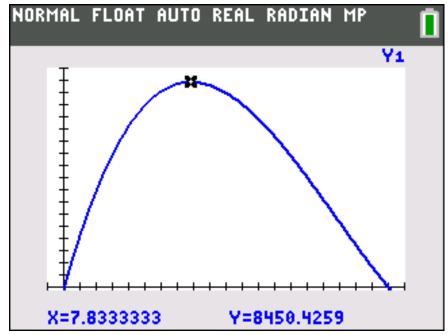
- Analyze a scale drawing of a rectangular box to determine the lengths, area, and volume of the box.
- Solve a real-world problem to optimize the volume of a box.
- Graph and analyze a function.

Vocabulary

- function
- maximum

Teacher Preparation and Notes

- Students will find the size of the square cut from the metal sheet to produce the maximum volume for the open box. If this is the first time your students are seeing this problem, it would be helpful to construct a few of these open-topped boxes and find their volumes. Have student construct a table for the box created when 1 cm, 2 cm, etc., have been cut out to make the height of the box. Having an idea of possible volumes will help with the window settings later on in the activity and explain why the graph increases then decreases.



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus C Silver Edition. It is also appropriate for use with the TI-84 Plus family with the latest TI-84 Plus operating system (2.55MP) featuring MathPrint ™ functionality. Slight variations to these directions given within may be required if using other calculator models.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials
- Any required calculator files can be distributed to students via handheld-to-handheld transfer.

Compatible Devices:

- TI-84 Plus Family
- TI-84 Plus C Silver Edition

Associated Materials:

- Creating_Boxes_Student.pdf
- Creating_Boxes_Student.doc

Introduction

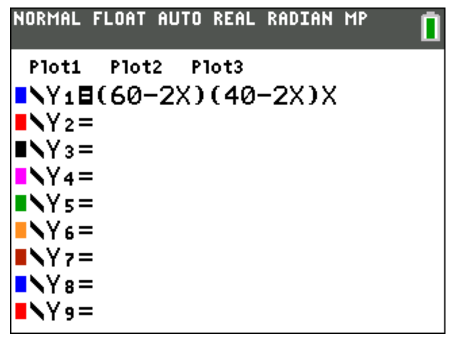
Introduce the box problem on the first page of the student worksheet. Explain to students to graph a relationship on a graphing calculator, it is necessary to carry out two tasks:

a. Define the relationship in terms of two variables, *x* and *y*.

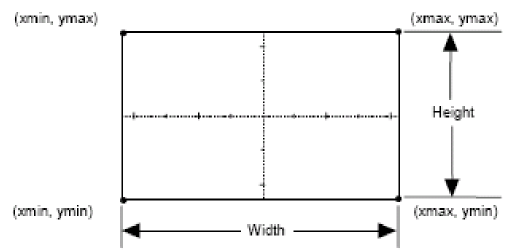
b. Define the portion of the coordinate plane over which you wish to view the graph.

Question 1

Students will enter the relationship $Y_1 = (60–2x)(40–2x)x$ for the volume in the [Y=] screen. The keystrokes to do this are [(] 60 [−] 2 [X,T,Θ,*n*] [)] [(] 40 [−] 2 [X,T,Θ,*n*] [)] [X,T,Θ,*n*].



You should discuss where this equation for the volume of the box comes from. The *x* represents the size of the side of the square which is cut and removed. It is also the height of the box when the sides are folded up. The (60 – 2*x*) represents the length of the box, which is 60 – 2 squares removed. The (40 – 2*x*) represents the width, which is 40 – 2 squares removed.



Viewing Window

1. Why must *x* have a value between 0 and 20?

<u>Answer</u>: Because 1 centimeter is the smallest whole centimeter cut and if more than 19 centimeters are cut, the box would have no volume (one of its dimensions becomes less than or equal to zero.)

Question 2

When students press [WINDOW], they will see a screen that allows them to define **Xmin**, **Xmax**, **Xscl**, **Ymin**, **Ymax**, and **Yscl**. Explain to students that the values of **Xmin** and **Xmax** define the left and right endpoints of the viewing window and the values of **Ymin** and **Ymax** define the upper and lower limits of the viewing window. The **Xscl** and **Yscl** determine the scale or size of the tick marks. For more details see the next Teacher Tip.

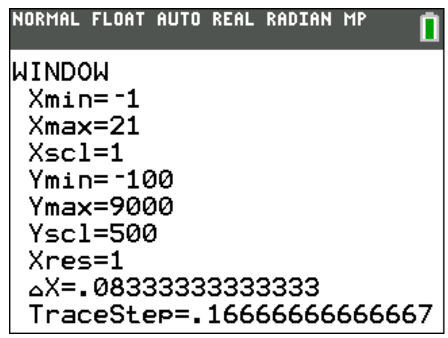
Xres can be changed to larger integer values to make the equation graph more quickly. This X resolution value can remain 1 for purposes of this activity.

Teacher Tip: Discuss with students that the values of **Xscl** and **Yscl** (abbreviations for *x-scale* and *y-scale*) have no effect on the window limits or on the appearance of the graph. They are used to define the distance between reference (or *tick*) marks that will appear on the two axes or along the left and bottom edges of any generated graph. Later in the activity when students graph the function, they can change the values for **Xscl** and **Yscl** to verify this and to help them understand why changing the scale does not change the appearance of the graph. It will be worth the time to let some students explore what happens when **Xscl** is changed. Some will think the graph should shrink or enlarge. Let them think about why this does not happen. Avoid the temptation to just tell them. Suggest that they think of a ruler. Does the distance between decimeters change when centimeter or millimeters marks are added?

Once students understand why the ranges for *x* and *y* have been chosen, they can enter the values given on the worksheet in the WINDOW screen.

Xmin = –1 Xmax = 21 Xscl = 1

Ymin = –100 Ymax = 9000 Yscl = 500

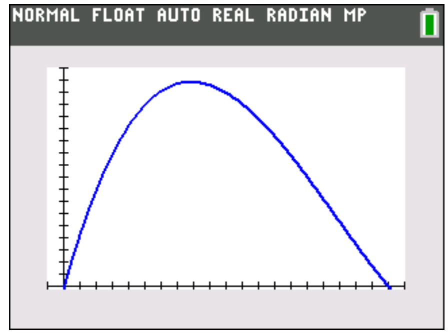


Teacher Tip: Defining the limits for the viewing windows for graphs provides students much needed practice in thinking about the function and its connection to the problem situation. They need to ask themselves, “What input values make sense for this problem?” (a restricted domain) and, “What output values do I get with those inputs?” (the range). When drawing graphs by hand, students will often just draw the two perpendicular axes, make tick marks, each 1 unit apart, and then find corresponding *y*-values to *x* = 0, 1, 2, and so on, even if it doesn’t make sense to use those values for the problem situation.

It makes sense in this problem to define **Xmin** to be 0 (or slightly less than 0 to allow for viewing near 0) and **Xmax** at 20 (or slightly more than 20 to allow for viewing near 20). The largest square that can be cut from the four corners is 20. If tick marks are desired, student can set **Xscl** to 1 and have tick marks at every unit along the *x*-axis.

Defining the limits on *y* is a little more challenging. A given value of **Y1** represents the volume of a box for some height *x*. Volume cannot be negative, so you can define **Ymin** to be slightly less than 0. You are looking for the *x* that would produce the largest value of *y* possible. This doesn’t give much of a hint about how large *y* can get. However, students’ previous work with tables showed that all volumes were less than 8500 cubic centimeters, so a reasonable value for **Ymax** is 9000. Since the height of this view is very large, the **Yscl** should be large. Enter 500 for **Yscl** so that tick marks on the *y*-axis are placed at 500, 1000, 1500, …, 9000.

Since the tasks of defining the expression and setting the window limits is completed, it is time to view the graph.



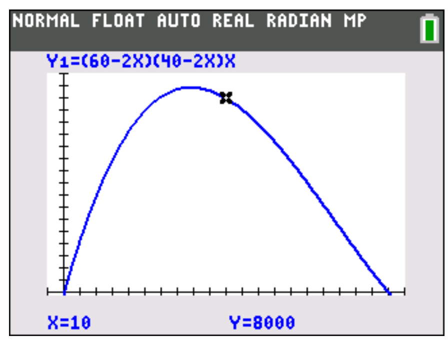
When students press [GRAPH], they should see a graph similar to the one at the right.

Starting with $x = 0$, the volumes increase (fairly quickly) up to a maximum volume between $x = 7$ and $x = 8$. The volumes then decrease beyond $x = 8$. The graph shows this pattern of increasing and decreasing also with the maximum occurring between $x = 7$ and $x = 8$.

2. In your group, write a brief explanation of why the graph should look the way it does based on your previous work with the volume table and the patterns you observed there.

   <u>Sample Answer:</u> The volume of the box increases to a maximum volume with *x* between 7 and 8, and then the volume decreases beyond this value of *x*. It makes sense that there would be a maximum near the center of the *x* values because a very small *x* value means that the height of the box is very small, and a very large *x* value means that the base of the box is very small; an *x* value in the central range would give the box the maximum height and base to optimize the volume.

Question 3



When students press [TRACE], the cursor should automatically be on the point (10, 8000). The coordinates of the blinking point are displayed at the bottom of the screen.

The equation graphed is displayed at the top of the screen.

You may want to ask students why the graph on the calculator is "flat" at the top. Are there several *x*-values that produce a maximum volume?

3. Do the coordinates of points on the graph agree with the values you computed earlier in this activity? Explain what these two values mean with respect to the box volume problem.

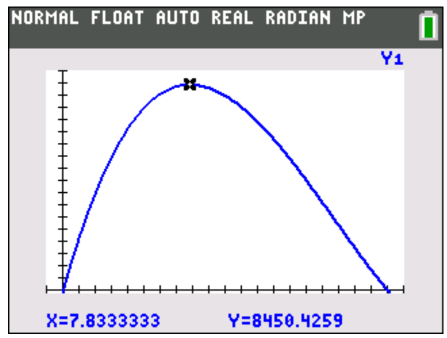
   Sample Answer: The coordinates are (10, 8000). The *x*-value of 10 cm indicates the side lengths in centimeters of the cutout square and the *y*-value of 8000 cm^3 is the volume of the corresponding box.

Question 4

Students can use the left and right arrow keys (◄ and ►) to see the coordinate of other points on the graph. Explain to students that the crosshair will move only along the graph and not just anywhere in the coordinate plane since [TRACE] is being used.

Discuss with students what they notice as they press the arrow keys. They should notice that the coordinates at the bottom of the screen change. Explain that the *x*-coordinates are determined by the calculator and are based upon the current values of **Xmin** and **Xmax**. The *y*-coordinates are computed using the expression $(60 – 2x)(40 – 2x)x$. Students will probably see that the *x*-values are not always "nice" numbers like those when tables of functions are generated. This sometimes makes it more difficult to decide on solutions to real world problems being modeled by the graphs.

Students are to try to locate the highest point on the graph. Note: On the TI-84 Plus the equation of the graph covers the top of the graph; students can press [2nd] [FORMAT] and select **ExprOff**. Then they should press [GRAPH] to return to the graph and press [TRACE] to view the coordinates.



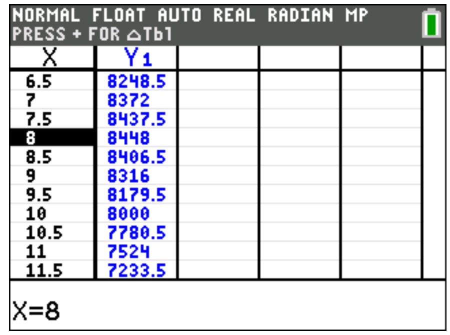
Teacher Tip: Discuss with students that not all of the decimal places shown in the coordinates of this point are meaningful. In fact, this point may not be the one students are looking for; it is just the closest point on this view of the graph. By examining the *x*-coordinates of the points on *each* side of this point, students can find an interval that contains the best value of *x*.

4. What are the values of *x* and *y* at the highest point on the graph?

Answer: Answers may vary, but will be around $x = 7.8474959$ and $y = 8450.4472$. On the TI-84 plus C, students will arrow over to find $x = 7.8333333$, $y = 8450.4259$

Question 5

Students are to use the arrow keys to determine the *x*-coordinates of the points immediately to the left and to the right of the point found in Question 4 and record them on their worksheet.



Explain to students that the solution to the problem lies somewhere between these two *x*-values. Students should check to see that the tabular solutions fall somewhere between the two values they have written for Question 5.

Have students press [2nd][TBLSET] to set up the Start and increment values.

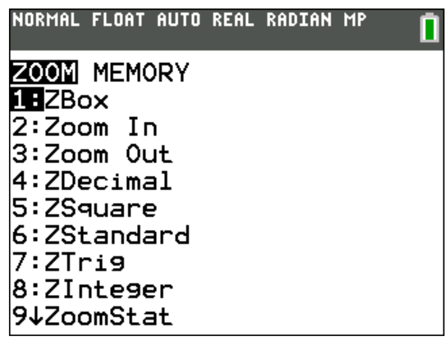
Have students press [2nd] [TABLE] to access the table of numbers.

5. Record the *x*-coordinates.

Answer: On the TI-84 Plus C the *x* value of the point to the left = 7.66666667, and *x* right = 8.

Question 6 and 7

Discuss with students that it is easy to examine the volume function over a smaller interval in order to obtain a more precise approximation to the desired solution. This can be accomplished a number of ways, but the fastest is to use the [ZOOM] capabilities of the calculator.

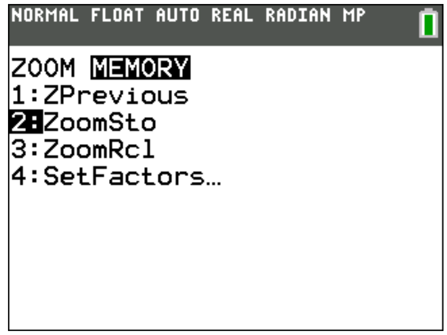


Examine the [ZOOM] menu with students. Two of the options shown are **Zoom In** and **Zoom Out**.

Zooming in is similar to bringing a portion of your current view closer to you for a finer examination—much like looking through a pair of binoculars. *Zooming out* is somewhat equivalent to enlarging your field of view so that you see more but in less detail.

Tech Tip: Since students need a closer, more detailed look at the points near the apparent maximum of the graph, they must zoom in around the region containing that maximum. A quick way to zoom in is to use the option named **ZBox**. **ZBox** lets you use the cursor to select opposite corners of a box to define the portion of the current view that you wish to enlarge.

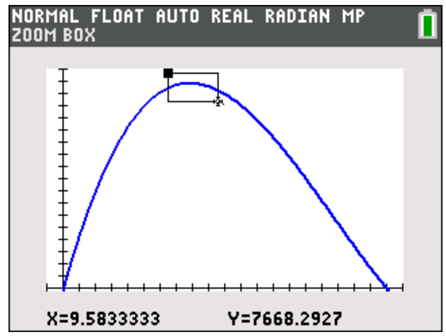
It is possible to save the current graph prior to zooming. That way if the students are not satisfied with their “zoom”, they can easily start over again. Once they press [ZOOM], have them arrow over to **MEMORY** and select the **ZoomSto** option. Then have students press [ZOOM] again and continue with the instructions given. Whenever they want to return to the stored zoom, they need to press [ZOOM], **MEMORY**, and select **ZoomRcl**.



When students hold down the arrow key as they create the box, the box disappears until they stop pressing the key.

Directions are given on the student worksheet for how students should select the box. Essentially they will need to press [ENTER] at the location for the upper left corner and press [ENTER] at the location for the lower right corner.

When student press [ENTER] the second time, the calculator should redraw the graph in the window they have just defined.



6. Record the coordinates of your estimated maximum value.

Answer: Answers will vary but should be around x = 7.847495 and y = 8450.4472.

7. By observing the x-coordinates of the points on each side of this point, you can again determine an interval that contains the desired solution. What x interval contains the value associated with the maximum height of the graph?

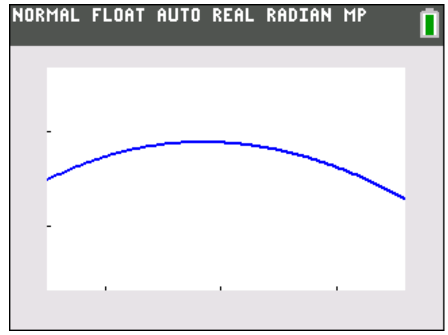
Answer: x interval: (7.83, 7.86). Answers will vary depending on the size of the box in the ZBox**.**

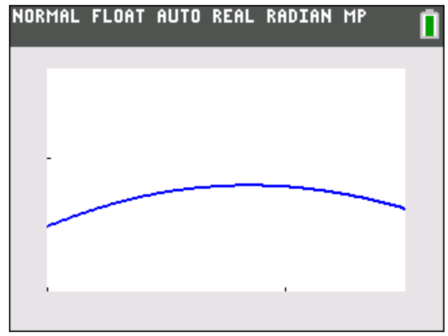
Question 8–10

If the tenths position in the x-interval is not the same for both values, or if both x-values when rounded to the nearest tenth are not the same, then the best choice for x we can make has to be a whole number. If students zoom one more time on the place where they believe the maximum volume will be, they will be able to provide a value for x that is accurate to the nearest tenth.

Students are now learning a bit about false precision and accuracy. To ensure that the x-value chosen is accurate to the nearest tenth, we need to find an interval of values where the tenths digits are the same when any of the x-values are rounded to the nearest tenth.

Students are to press [ZOOM] and select **Zbox** again. They should make another box around the area on the graph where they think the maximum volume lies. They should also press [TRACE] to find new *x*-values for an interval containing the maximum volume as they have done previously.





Students may have to do several zooms in order to get values in an *x* interval whose tenths digits round to the same number. If they zoom in with a tiny box around the maximum value, they may arrive at such an interval sooner, for example (7.846, 7.847).

Ask students if they notice how the graph appears to be getting flatter and flatter as one zooms in on the maximum value. Several interesting "upper level" mathematical ideas can spring from this investigation: the fact that every graph is "locally linear", that is, that all graphs will appear to be a straight line as we zoom in on smaller and smaller intervals of *x*, and that the slope of the tangent line at the local maxima and minima of a function is zero.

Remind the students that they cannot simply read off the volume from the graph as the volume is a computed value based on the *x*-value given. They need to compute the volume based on their chosen value for *x*.

8. Write the new interval for the maximum volume. Determine whether you can provide an *x*-value accurate to the nearest tenth. Remember, you want all numbers in your interval to round to the same tenth's value.

Answer: The *x*-value to the nearest tenth is 7.8 cm.

9. Based on the results from your trace of the graph, what size squares should the class cut from the pieces of sheet metal? What are the dimensions and volume of the box with the largest volume?

Answer: To the nearest tenth of a centimeter, Side length of square: 7.8 cm

H = 7.8 cm L = 44.4 cm W = 24.4 cm V = 8450.208 cm^3

10. Discuss in your groups the advantages of investigating the box volume problem graphically. Write those advantages below.

Answer: Advantages will vary based upon the group discussion. Some students prefer the numerical/tabular approach. The ΔTbl feature makes "zooming in" on a table fairly easy. Others prefer a picture and would rather zoom in on the graph for a maximum value. Sometimes when viewing a finite table of values, it is difficult to get a sense of how the function "behaves" over a larger interval; the graph provides a "bigger picture". One can find a maximum value fairly quickly on the table without having much sense about how the function behaves. When creating a graph, you have the additional challenge of defining the window. This requires some knowledge of the behavior of the function.