TI in Focus: AP® Calculus

2022 AP® Calculus Exam: AB2
Scoring Guidelines and Solutions

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Outline

(1) Free Response Question

(2) Point Distribution

(3) Solutions (using technology)

(4) Scoring Notes

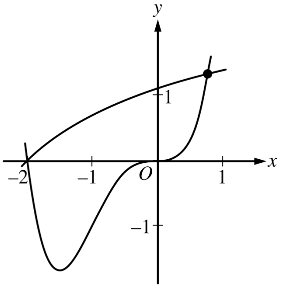
(5) Common Errors

(6) Problem Extensions

Part A (AB): Graphing calculator required
Question 2 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

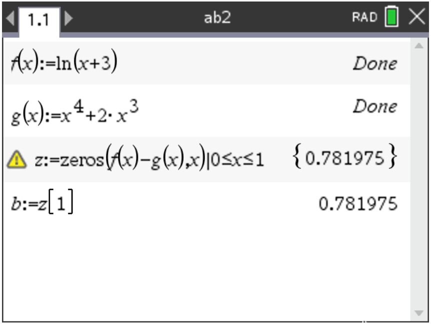
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

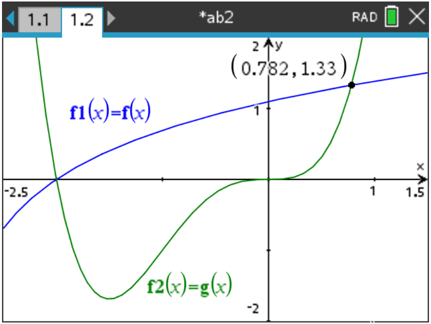


Let f and g be the functions defined by $f(x) = \ln(x+3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

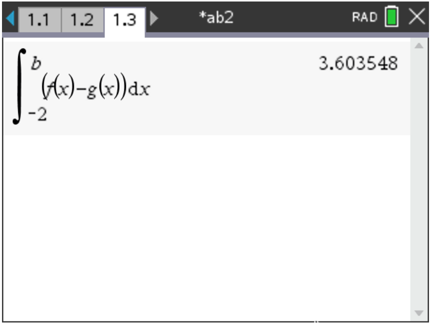
	Model Solution	Scoring	
(a)	Find the area of the region enclosed by the graphs of f and g.		
	$\ln(x+3) = x^4 + 2x^3 \Rightarrow x = -2,\ x = B = 0.781975$	Integrand	**1 point**
	$\int_{-2}^{B} (f(x) - g(x))\, dx = 3.603548$	Limits of integration	**1 point**
	The area of the region is 3.604 (or 3.603).	Answer	**1 point**

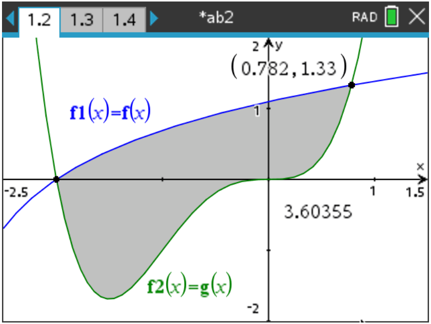
Solution





Solution (Continued)





$b = 0.781975$

$$\int_{-2}^{b} [f(x) - g(x)]\ dx = 3.604$$

Scoring notes:

- Other forms of the integrand in a definite integral, e.g., $|f(x)-g(x)|$, $|g(x)-f(x)|$, or $g(x)-f(x)$, earn the first point.

All of these integrands can be used to find the area of the region.

- To earn the second point, the response must have a lower limit of -2 and an upper limit expressed as either the letter B with no value attached, or a number that is correct to the number of digits presented, with at least one and up to three decimal places.
  - Case 1: If the response did not earn the second point because of an incorrect value of B, $0 < B < 1$, but used a lower limit of -2, the response earns the third point only for a consistent answer.

$$B = 0.9 \quad \Rightarrow \quad \int_{-2}^{0.9} [f(x) - g(x)]\, dx = 3.562$$

1 - 0 - 1

- Case 2: If the response did not earn the second point because the lower limit used was $x = 0$, but the response used a correct upper limit of B, the response earns the third point for a consistent answer of 0.708 (or 0.707).

$$B = 0.781975 \quad \Rightarrow \quad \int_0^B [f(x) - g(x)]\, dx = 0.708$$ 1 - 0 - 1

- Case 3: If a response uses any other incorrect limits it does not earn the second or third points.

$$B = 0.6 \quad \Rightarrow \quad \int_{-1}^B [f(x) - g(x)]\, dx = 1.845$$ 1 - 0 - 0

- A response containing the integrand $g(x) - f(x)$ must interpret the value of the resulting integral correctly to earn the third point. For example, the following response earns all 3 points: $\int_{-2}^{B}(g(x) - f(x))\,dx = -3.604$ so the area is 3.604. However, the response "Area $= \int_{-2}^{B}(g(x) - f(x))\,dx = 3.604$" presents an untrue statement and earns the first and second points, but not the third point.

$$B = 0.781975 \quad \Rightarrow \quad \int_{-2}^{B} [g(x) - f(x)\,dx = -3.604$$

Therefore, the area of the region is 3.604

- A response must earn the first point in order to be eligible for the third point. If the response has earned the second point, then only the correct answer will earn the third point.

- Instructions for scoring a response that presents an integrand of $\ln(x+3) - x^4 + 2x^3$ and the correct answer are shown in the "Global Special case" after part (d).

$$f(x) - g(x) = \ln(x+3) - (x^4 + 2x^3) = \ln(x+3) - x^4 + 2x^3$$

Global Special case: A response may incorrectly simplify $f(x) - g(x)$ to $j(x) = \ln(x+3) - x^4 + 2x^3$ instead of $\ln(x+3) - x^4 - 2x^3$. Because this question is calculator active, a response with this incorrect simplification may nevertheless present correct answers.

- In any part of the question, a response that starts correctly by using $f(x) - g(x)$, then presents $j(x)$, is eligible for all points in that part.
- The first time a response implicitly presents $f(x) - g(x)$ as $j(x) = \ln(x+3) - x^4 + 2x^3$ (with no explicit connection) in any part of this question, the response loses a point. The response is then eligible for all remaining points for a correct or consistent answer.
- In part (a), the consistent answer using $j(x)$ is negative and will not earn the third point.
- In part (b), the consistent answer using $j(x)$ is that $j'(-0.5) = 2.4 > 0$, so h is increasing at $x = -0.5$.
- In part (c), the consistent answer using $j(x)$ is 252.187 (or 252.188).
- In part (d), the consistent answer using $j(x)$ is 20.287.

Common Errors

(1) There were very few responses that actually reversed the integrand, using $g(x) - f(x)$ instead of $f(x) - g(x)$.

(2) A few responses divided the area of the region into multiple pieces, and used definite integrals to find the area of each piece separately.

(3) Many responses did not present the upper limit of integration correctly.

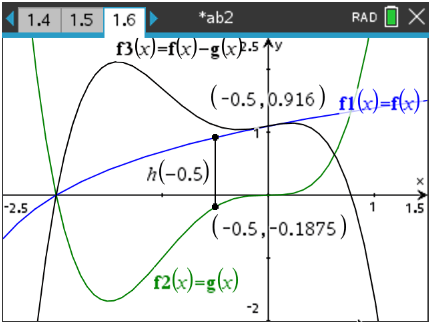
Write the x-value with at least 3 digits of accuracy.

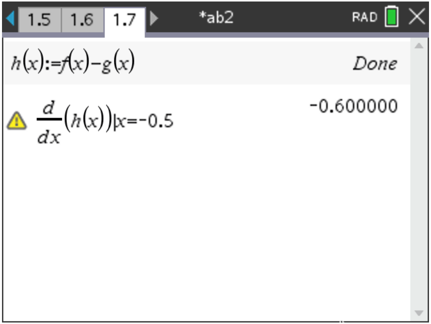
Store the x-value using the greatest degree of accuracy allowed by the technology.

(b) For $-2 \le x \le B$, let $h(x)$ be the vertical distance between the graphs of f and g. Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.

$h(x) = f(x) - g(x)$ $h'(x) = f'(x) - g'(x)$ $h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6$ (or -0.599)	Considers $h'(-0.5)$ **1 point** – OR – $f'(x) - g'(x)$
Since $h'(-0.5) < 0$, h is decreasing at $x = -0.5$.	Answer with reason **1 point**

Solution





$h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x)$

$h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6 < 0$

Therefore, h is decreasing at $x = -0.5$.

Scoring notes:

- The response need not present the value of $h'(-0.5)$. The last line earns both points. However, if a value is presented it must be correct for the digits reported up to three decimal places.

The question did not ask for the value of $h'(-0.5)$.
Therefore, the sign of $h'(-0.5)$ is sufficient.

- A response that reports an incorrect value of $h'(-0.5)$ earns only the first point.

This response has considered $h'(-0.5)$ but does not earn the second point.

- A response that presents only $h'(x)$ does not earn either point.

This response has not considered $h'(-0.5)$.

- The only response that earns the second point for concluding "h is increasing" is described in the Global Special case provided after part (d).

This is the case in which the response incorrectly simplifies $f(x) - g(x)$.

- A response that compares the values of $f'(x)$ and $g'(x)$ at $x = -0.5$ earns the first point and is eligible for the second point. This comparison can be made symbolically or verbally; for example, the response "the rate of change of $f(x)$ is less than the rate of change of $g(x)$ at $x = -0.5$" earns the first point.

$f'(-0.5) = 0.4 \qquad g'(-0.5) = 1.0$

Since $f'(-0.5) < g'(-0.5)$ h is decreasing at $x = -0.5$.

Common Errors

(1) Most responses interpreted vertical distance as $h(x) = f(x) - g(x)$. But several did not use this notation $h(x)$ in their explanation.

(2) In comparing $f'(-0.5)$ and $g'(-0.5)$ verbally: errors in explanation.

(c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x-axis are squares. Find the volume of the solid.

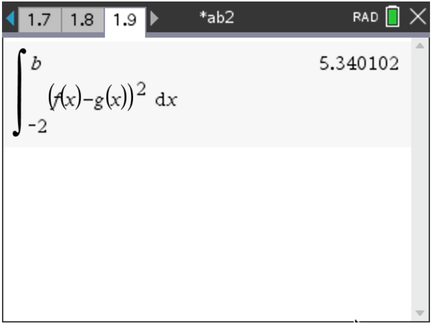
$\int_{-2}^{B}(f(x)-g(x))^2\,dx = 5.340102$	Integrand	**1 point**
The volume of the solid is 5.340.	Answer	**1 point**

Solution

$$A(x) = s^2 = [f(x) - g(x)]^2$$

$$V = \int_{-2}^{b} [f(x) - g(x)]^2\,dx = 5.340102$$

The volume of the solid is 5.34



Scoring notes:

- The first point is earned for an integrand of $k(f(x)-g(x))^2$ or its equivalent with $k \neq 0$ in any definite integral. If $k \neq 1$, then the response is not eligible for the second point.

- A response that does not earn the first point is ineligible to earn the second point, with the following exceptions:
  - A response which has a presentation error in the integrand (for example, mismatched or missing parentheses, misplaced exponent) does not earn the first point, but would earn the second point for the correct answer. A response which has a presentation error in the integrand and which reports an incorrect answer earns no points.
  - A response that presents an integrand of $\left(\ln(x+3)-x^4+2x^3\right)^2$. Scoring instructions for this case are provided in the Global Special case after part (d).

$$\int_{-2}^{B} (f(x)-g(x)^2\,dx = 5.34$$

- A response that uses incorrect limits is only eligible for the second point provided the limits are imported from part (a) in case 1 or case 2. In both of these situations, the second point is earned only for answers consistent with the imported limits.

Common Errors

(1) Area of a cross section as the area of a circle.

(2) Area of a cross section as $f(x)^2 - g(x)^2$.

(3) Presentation errors and simplification errors in the integrand.

(4) If a correct definite integral, then most responses evaluated correctly.

(d) A vertical line in the xy-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

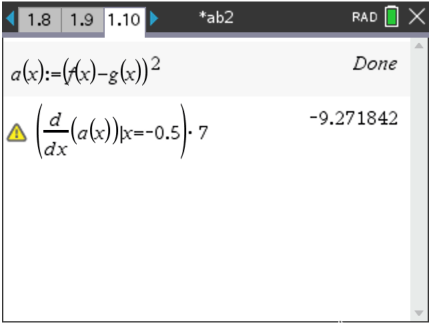
The cross section has area $A(x) = (f(x) - g(x))^2$. $\frac{d}{dt}[A(x)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$	$\frac{dA}{dx} \cdot \frac{dx}{dt}$	**1 point**
$\frac{d}{dt}[A(x)]\Big\vert_{x=-0.5} = A'(-0.5) \cdot 7 = -9.271842$ At $x = -0.5$, the area of the cross section above the line is changing at a rate of -9.272 (or -9.271) square units per second.	Answer	**1 point**

Solution

$$A(x) = [f(x) - g(x)]^2$$

$$\frac{d}{dt}[A(t)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

$$= A'(-0.5) \cdot 7 = -9.271842$$



At $x = -0.5$, the area of the cross section above the line is changing at a rate of -9.272 (or -9.271) square units per second.

Scoring notes:

- The first point may be earned by presenting $\frac{dA}{dx} \cdot \frac{dx}{dt}$, $A' \cdot \frac{dx}{dt}$, $A'(x) \cdot \frac{dx}{dt}$, $A'(x) \cdot x'$, $A'(-0.5) \cdot 7$, or $k \cdot 7$, where k is a declared value of $A'(-0.5)$, or any equivalent expression, including $2(f(x) - g(x))(f'(x) - g'(x))\frac{dx}{dt}$.

- If a response defines $f(x) - g(x)$ as a function in parts (b) or (c) (for example, $h(x) = f(x) - g(x)$), then a correct expression for $\frac{dA}{dt}$ (for example, $2h\frac{dh}{dt}$) earns the first point.

- A response that imports a function $A(x)$ declared in part (c) is eligible for both points (the answer must be consistent with the imported function $A(x)$).

- A response that presents an incorrect function for $A(x)$ that is not imported from part (c) is eligible only for the first point.

- Except when $A(x)$ is imported from part (c), the second point is earned only for the correct answer.

- A response that does not earn the first point is ineligible to earn the second point except in the special case noted below.

Common Errors

(1) Most responses were able to interpret the vertical line moving at a constant rate of 7 units per second as $\frac{dx}{dt}$.

(2) Many responses did not convey an understanding that the cross-sectional area was also a function of t, and therefore, that the Chain Rule had to be used to compute $\frac{dA}{dt}$

(3) Many responses expressed the cross-section areas as the square of another variable and used the Chain Rule.

$$\frac{dA}{dt} = 2s\,\frac{ds}{dt}$$

But did not express s correctly.

Additional Questions

(1) Find the total area bounded by the graphs of f and g between $x = -2.5$ and $x = 1.5$.

(2) Find the value a, $-2 \leq a \leq 0$, such that $f'(a) = g'(a)$.

(3) Find an equation of the line tangent l to the graph of f at the point where $x = -2.9$.

(4) Find the area of the region bounded by the tangent line l and the graphs of f and g.

(5) Let R be the region enclosed by the graphs of f and g. Find the volume of the solid generated by revolving the region R about the line $y = -2$.

(6) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x-axis are semicircles with diameter in the base. Find the volume of the solid.

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